Chapter 5

Analysis of SISO Control Loops

Topics to be covered

For a given controller and plant connected in feedback we ask and answer the following questions:

- Is the loop stable?
- What are the sensitivities to various disturbances?
- What is the impact of linear modeling errors?
- How do small nonlinearities impact on the loop?

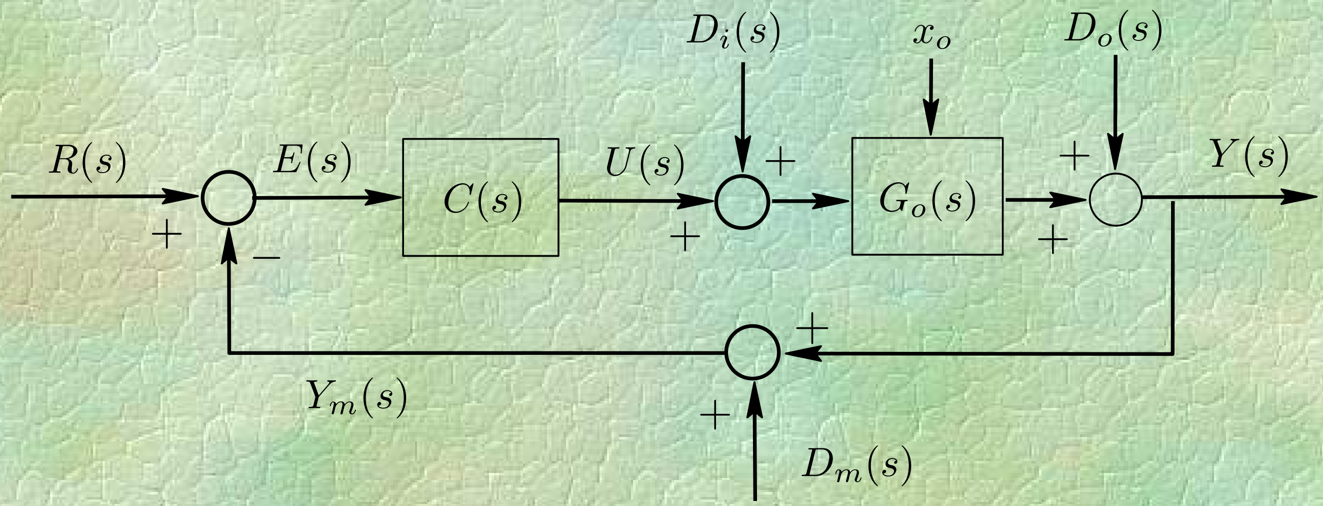
We also introduce several analysis tools; specifically

- Root locus
- Nyquist stability analysis

Feedback Structures

We will see that feedback can have many desirable properties such as the capacity to reduce the effect of disturbances, to decrease sensitivity to model errors or to stabilize an unstable system. We will also see, however, that ill-applied feedback can make a previously stable system unstable, add oscillatory behaviour into a previously smooth response or result in high sensitivity to measurement noise.

Figure 5.1: *Simple feedback control system*



In the loop shown in Figure 5.1 we use transfer functions and Laplace transforms to describe the relationships between signals in the loop. In particular, $C(s)$ and $G_0(s)$ denote the transfer functions of the controller and the nominal plant model respectively, which can be represented in fractional form as:

$$C(s) = \frac{P(s)}{L(s)}$$

$$G_o(s) = \frac{B_o(s)}{A_o(s)}$$

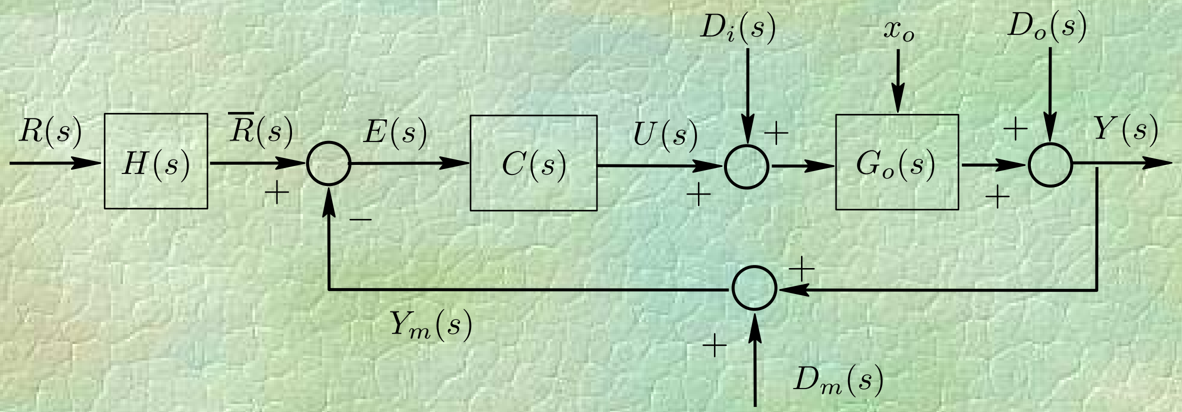
Laplace Transforms of System Input and Output

$$U(s) = \frac{C(s)}{1 + G_o(s)C(s)}\left(R(s) - D_m(s) - D_o(s) - G_o(s)D_i(s) - \frac{f(s, x_o)}{A(s)}\right)$$

and

$$Y(s) = \frac{1}{1 + G_o(s)C(s)}\left[G_o(s)C(s)(R(s) - D_m(s)) + D_o(s) + G_o(s)D_i(s) + \frac{f(s, x_o)}{A(s)}\right]$$

Figure 5.2: *Two degree of freedom closed loop*



$$Y(s) = \frac{G_o(s)C(s)H(s)}{1+G_o(s)C(s)}R(s) + \frac{1}{1+G_o(s)C(s)}\left(D_o(s) + \frac{f(s,x_o)}{A_o(s)}\right) + \frac{G_o(s)}{1+G_o(s)C(s)}D_i(s) - \frac{G_o(s)C(s)}{1+G_o(s)C(s)}D_m(s)$$

Nominal Sensitivity Functions

$$T_o(s) \triangleq \frac{G_o(s)C(s)}{1 + G_o(s)C(s)} = \frac{B_o(s)P(s)}{A_o(s)L(s) + B_o(s)P(s)}$$

$$S_o(s) \triangleq \frac{1}{1 + G_o(s)C(s)} = \frac{A_o(s)L(s)}{A_o(s)L(s) + B_o(s)P(s)}$$

$$S_{io}(s) \triangleq \frac{G_o(s)}{1 + G_o(s)C(s)} = \frac{B_o(s)L(s)}{A_o(s)L(s) + B_o(s)P(s)}$$

$$S_{uo}(s) \triangleq \frac{C(s)}{1 + G_o(s)C(s)} = \frac{A_o(s)P(s)}{A_o(s)L(s) + B_o(s)P(s)}$$

These functions are given specific names as follows:

$T_0(s)$	:	Nominal complementary sensitivity
$S_0(t)$	:	Nominal sensitivity
$S_{i0}(s)$	:	Nominal input disturbance sensitivity
$S_{u0}(s)$	:	Nominal control sensitivity

Relationship between sensitivities

$$S_o(s) + T_o(s) = 1$$

$$S_{io}(s) = S_o(s)G_o(s) = \frac{T_o(s)}{C(s)}$$

$$S_{uo}(s) = S_o(s)C(s) = \frac{T_o(s)}{G_o(s)}$$

Internal Stability

Definition 5.1 (Internal stability). We say that the nominal loop is internally stable if and only if all eight transfer functions in the equation below are stable.

$$\begin{bmatrix} Y_o(s) \\ U_o(s) \end{bmatrix} = \frac{\begin{bmatrix} G_o(s)C(s) & G_o(s) & 1 & -G_o(s)C(s) \\ C(s) & -G_o(s)C(s) & -C(s) & -C(s) \end{bmatrix}}{1 + G_o(s)C(s)} \begin{bmatrix} H(s)R(s) \\ D_i(s) \\ D_o(s) \\ D_m(s) \end{bmatrix}$$

Link to Characteristic Equation

Lemma 5.1 (Nominal internal stability)

Consider the nominal closed loop depicted in Figure 5.2. Then the nominal closed loop is internally stable if and only if the roots of the nominal closed loop characteristic equation

$$A_o(s)L(s) + B_o(s)P(s) = 0$$

all lie in the open left half plane. We call $A_0L + B_0P$ the nominal closed-loop characteristic polynomial.

Stability and Polynomial Analysis

Consider a polynomial of the following form:

$$p(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$$

The problem to be studied deals with the question of whether that polynomial has any root with nonnegative real part. Obviously, this equation can be answered by *computing the n roots* of $p(s)$. However, in many applications it is of special interest to study the interplay between the location of the roots and certain polynomial coefficients.

Some Polynomial Properties of Special Interest

Property 1: The coefficient a_{n-1} satisfies

$$a_{n-1} = -\sum_{i=1}^{n} \lambda_i$$

Property 2: The coefficient a_0 satisfies

$$a_0 = (-1)^n \prod_{i=1}^{n} \lambda_i$$

Property 3: If all roots of $p(s)$ have negative real parts, it is necessary that $a_i > 0$, $i \in \{0, 1, \ldots, n-1\}$.

Property 4: If any of the polynomial coefficients is nonpositive (negative or zero), then, one or more of the roots have nonnegative real plant.

Routh's Algorithm

$$p(s) = \sum_{i=0}^{n} a_i s^i$$

The Routh's algorithm is based on the following numerical array:

$$\begin{array}{llllll}
\mathbf{s^n} & \gamma_{0,1} & \gamma_{0,2} & \gamma_{0,3} & \gamma_{0,4} & \cdots \\
\mathbf{s^{n-1}} & \gamma_{1,1} & \gamma_{1,2} & \gamma_{1,3} & \gamma_{1,4} & \cdots \\
\mathbf{s^{n-2}} & \gamma_{2,1} & \gamma_{2,2} & \gamma_{2,3} & \gamma_{2,4} & \cdots \\
\mathbf{s^{n-3}} & \gamma_{3,1} & \gamma_{3,2} & \gamma_{3,3} & \gamma_{3,4} & \cdots \\
\mathbf{s^{n-4}} & \gamma_{4,1} & \gamma_{4,2} & \gamma_{4,,3} & \gamma_{4,4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\mathbf{s^2} & \gamma_{n-2,1} & \gamma_{n-2,2} & & & \\
\mathbf{s^1} & \gamma_{n-1,1} & & & & \\
\mathbf{s^0} & \gamma_{n,1} & & & &
\end{array}$$

Table 5.1: *Routh's array*

Where

$$\gamma_{0,i} = a_{n+2-2i}; \quad i = 1, 2, \ldots, m_0 \qquad \text{and} \quad \gamma_{1,i} = a_{n+1-2i}; \quad i = 1, 2, \ldots, m_1$$

with $m_0 = (n+2)/2$ and $m_1 = m_0$-1 for n even and $m_1 = m_0$ for n odd. Note that the elements $\gamma_{0,i}$ and $\gamma_{1,i}$ are the coefficients of the polynomials arranged in alternated form. Furthermore

$$\gamma_{k,j} = \frac{\gamma_{k-1,1}\,\gamma_{k-2,j+1} - \gamma_{k-2,1}\,\gamma_{k-1,j+1}}{\gamma_{k-1,1}}\;; \qquad k = 2, \ldots, n \qquad j = 1, 2, \ldots, m_j$$

Result:

Consider a polynomial $p(s)$ given by (5.5.8) and its associated array as in Table 5.1. Then the number of roots with real part greater than zero is equal to the number of sign changes in the first column of the array.

Root Locus (RL)

Another classical tool used to study stability of equations of the type given above is root locus. The root locus approach can be used to examine the location of the roots of the characteristic polynomial as one parameter is varied.

Consider the following equation

$$1 + \lambda F(s) = 0 \qquad \text{where} \qquad F(s) = \frac{M(s)}{D(s)}$$

with $\lambda \geq 0$ and *M*, *N* have degree *m*, *n* respectively.

Root locus building rules include:

R1 The number of roots of the equation $(1 + \lambda F(s) = 0)$ is equal to $\max\{m,n\}$. Thus, the root locus has $\max\{m,n\}$ branches.

R2 From $(1 + \lambda F(s) = 0)$ we observe that s_0 belongs to the root locus (for $\lambda \geq 0$) if and only if

$$\arg F(s_0) = (2k+1)\pi \qquad \text{for} \qquad k \in \mathbb{Z}.$$

R3 From equation $(1 + \lambda F(s) = 0)$ we observe that if s_0 belongs to the root locus, the corresponding value of λ is λ_0 where

$$\lambda_0 = \frac{-1}{F(s_0)}$$

R4 A point s_0 on the real axis, i.e. $s_0 \in \mathbb{R}$, is part of the root locus (for $\lambda \geq 0$), if and only if, it is located to the left of an odd number of poles and zeros (so that R2 is satisfied).

R5 When λ is close to zero, then n of the roots are located at the poles of $F(s)$, i.e. at $p_1, p_2, \ldots, p_n$ and, if $n < m$, the other $m - n$ roots are located at ∞ (we will be more precise on this issue below).

R6 When λ is close to ∞, then m of these roots are located at the zeros of $F(s)$, i.e. at $c_1, c_2, \ldots, c_m$ and, if $n > m$, the other $n - m$ roots are located at ∞ (we will be more precise on this issue below).

R7 If $n > m$, and λ tends to ∞, then, $n - m$ roots asymptotically tend to ∞, following asymptotes which intersect at $(\sigma, 0)$, where

$$\sigma = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} c_i}{n-m}$$

The angles of these asymptotes are $\eta_1, \eta_2, \ldots, \eta_{m-n}$, where

$$\eta_k = \frac{(2k-1)\pi}{n-m}; \qquad k = 1, 2, \ldots, n-m$$

R8 If $n < m$, and λ tends to zero, then, m-n roots asymptotically tend to ∞, following asymptotes which intersect at $(\sigma, 0)$, where

$$\sigma = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} c_i}{m-n}$$

The angles of these asymptotes are $\eta_1, \eta_2, \ldots \eta_{m-n}$, where

$$\eta_k = \frac{(2k-1)\pi}{n-m}; \qquad k = 1, 2, \ldots, m-n$$

R9 When the root locus crosses the imaginary axis, say at $s = \pm jw_c$, then w_c can be computed either using the Routh Hurwitz algorithm, or using the fact that $s^2 + w_c^2$ divides exactly the polynomial $D(s) + \lambda M(s)$, for some positive real value of λ.

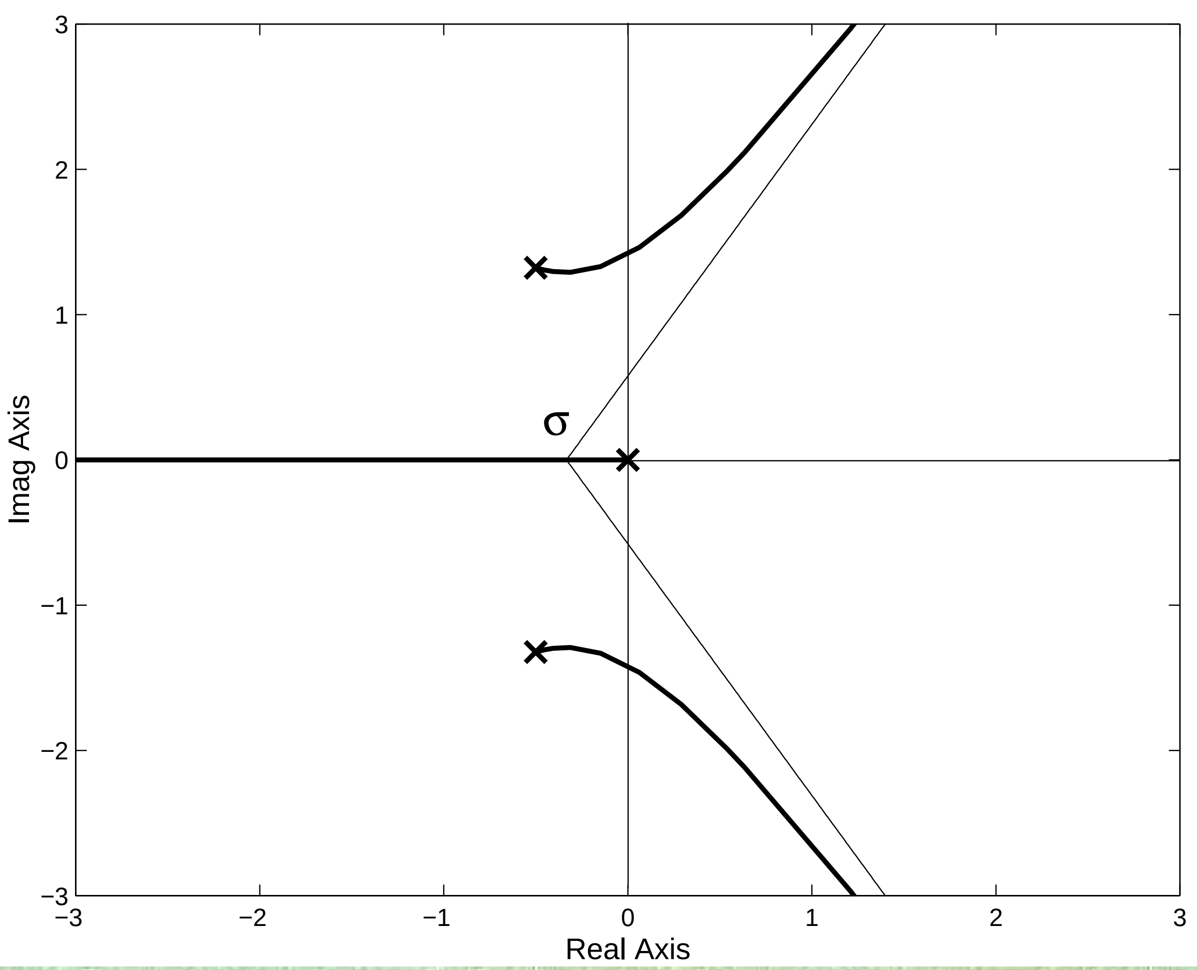
Example

Consider a plant with transfer function $G_0(s)$ and a feedback controller with transfer function $C(s)$, where

$$G_o(s) = \frac{1}{(s-1)(s+2)} \qquad \text{and} \qquad C(s) = 4\frac{s+\alpha}{s}$$

We want to know how the location of the closed loop poles change for α moving in $\mathbb{R}^+$.

Figure 5.3: *Locus for the closed loop poles when the controller zero varies*



Nominal Stability using Frequency Response

A classical and lasting tool that can be used to assess the stability of a feedback loop is Nyquist stability theory. In this approach, stability of the closed loop is predicted using the open loop frequency response of the system. This is achieved by plotting a polar diagram of the product $G_0(s)C(s)$ and then counting the number of encirclements of the (-1,0) point. We show how this works below.

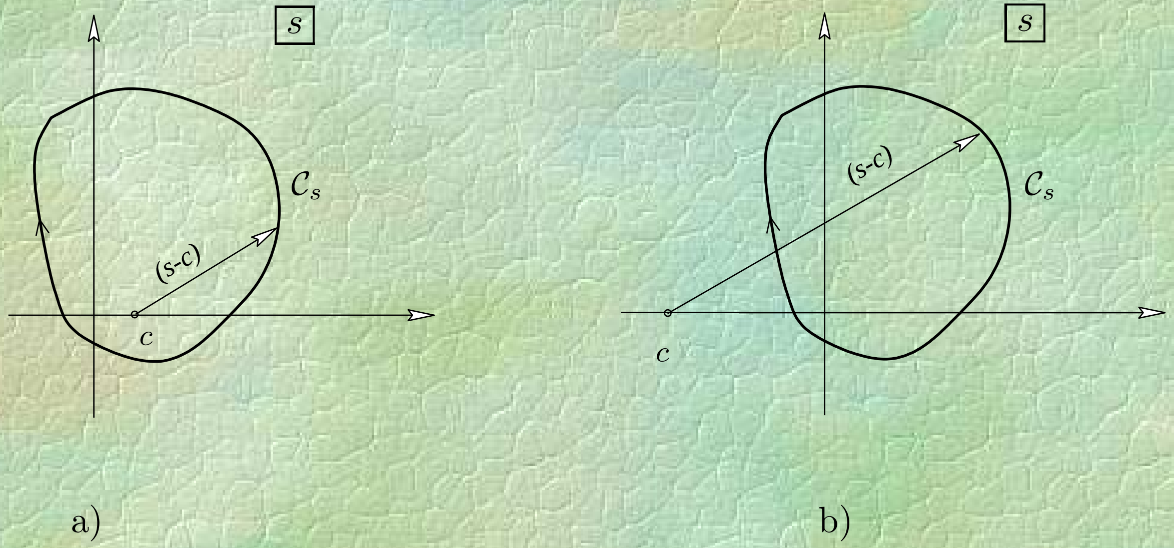
Nyquist Stability Analysis

The basic idea of Nyquist stability analysis is as follows:

assume you have a closed oriented curve C_s in $\boxed{s}$ which encircles Z zeros and P poles of the function $F(s)$. We assume that there are no poles on C_s.

If we move along the curve C_s in a defined direction, then the function $F(s)$ maps C_s into another oriented closed curve, C_F in $\boxed{F}$.

Illustration: *Single zero function and Nyquist path C_s in $\boxed{s}$*



c inside C_s

c outside C_s

Observations

Case (a): c inside C_s

We see that as s moves clockwise along C_s, the angle of $F(s)$ changes by -2π[rad], i.e. the curve C_F will enclose the origin in $\boxed{F}$ once in the clockwise direction.

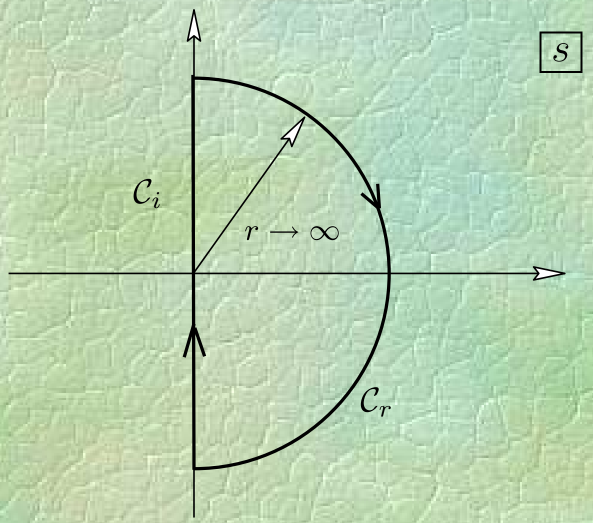
Case (b): c outside C_s

We see that as s moves clockwise along C_s, the angle of $F(s)$ changes by 0[rad], i.e. the curve C_F will enclose the origin in $\boxed{F}$ once in the clockwise direction.

More general result:

Consider a general function $F(s)$ and a closed curve $\mathcal{C}_s$ in $\boxed{s}$. Assume that $F(s)$ has Z zeros and P poles inside the region enclosed by $\mathcal{C}_s$. Then as s moves clockwise along $\mathcal{C}_s$, the resulting curve $\mathcal{C}_F$ encircles the origin in $\boxed{F}$ Z-P times in a clockwise direction.

To test for poles in the Right half Plane, we choose C_s as the following Nyquist path



As s traverses the Nyquist path in $\boxed{s}$, then we plot a polar plot of $F = G_0C$. Actually we shift the origin to "-1" so that encirclements of -1 count the zeros of $G_0C + 1$ in the right half plane.

Final Result

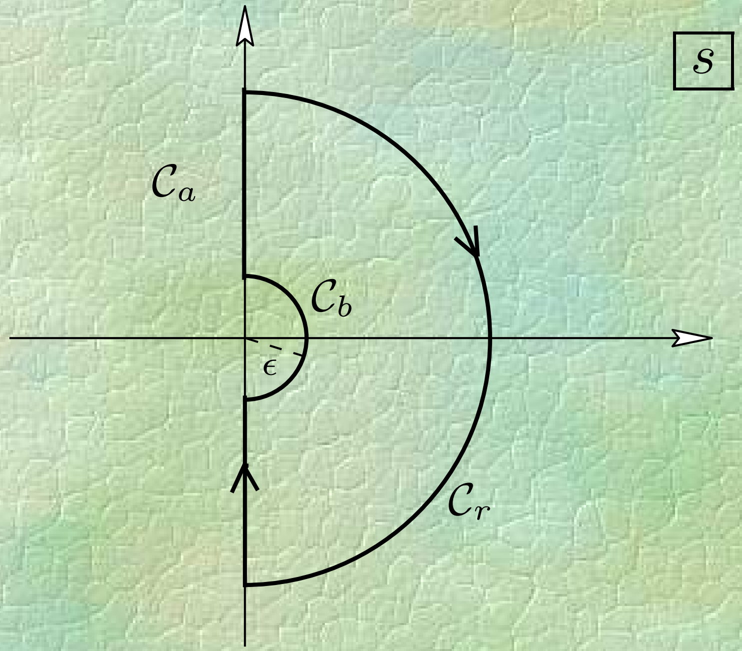
Theorem 5.1:

If a proper open loop transfer function $G_0(s)C(s)$ has P poles in the open RHP, and none on the imaginary axis, then the closed loop has Z poles in the open RHP if and only if the polar plot $G_0(s_w)C(s_w)$ encircles the point (-1,0) clockwise N=Z-P times.

Discussion

- If the system is open loop stable, then for the closed loop to be internally stable it is necessary and sufficient that no unstable cancellatgions occur and that the Nyquist plot of $G_0(s)C(s)$ *does not encircle the point* (-1,0).
- If the system is open loop unstable, with P poles in the open RHP, then for the closed loop to be internally stable it is necessary and sufficient that no unstable cancellations occur and that the Nyquist plot of $G_0(s)C(s)$ *encircles the point* (-1,0) *P times counterclockwise.*
- If the Nyquist plot of $G_0(s)C(s)$ passes exactly through the point (-1,0), there exists an $w_0 \in \mathbb{R}$ such that $F(jw_0) = 0$, i.e. the closed loop has poles located exactly on the imaginary axis. This situation is known as a *critical stability condition*.

Figure 5.6: *Modified Nyquist path (To account for open loop poles or zeros on the imaginary axis).*



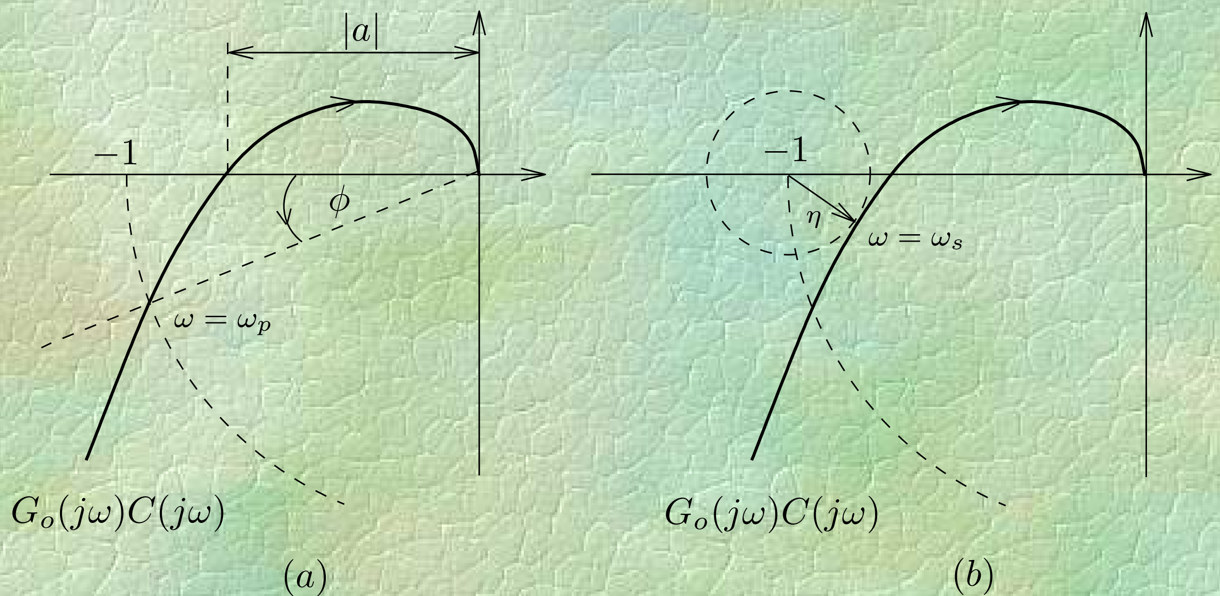
Theorem 5.2 (Nyquist theorem):

Given a proper open loop transfer function $G_0(s)C(s)$ with P poles in the open RHP, then the closed loop has Z poles in the open RHP if and only if the plot of $G_0(s)C(s)$ encircles the point (-1,0) clockwise N=Z-P times when s travels along the modified Nyquist path.

Relative Stability: Stability margins and Sensitivity Peaks

In control system design, one often needs to go beyond the issue of closed loop stability. In particular, it is usually desirable to obtain some quantitative measures of how far from instability the nominal loop is, i.e. to quantify relative stability. This is achieved by introducing measures which describe the distance from the nominal open loop frequency response to the critical stability point (-1,0).

Figure 5.7: *Stability margins and sensitivity peak*



Gain and Phase Margins *Peak Sensitivity*

(a) The gain margin, M_g, and the phase margin M_f are defined as follows (see Figure 5.7):

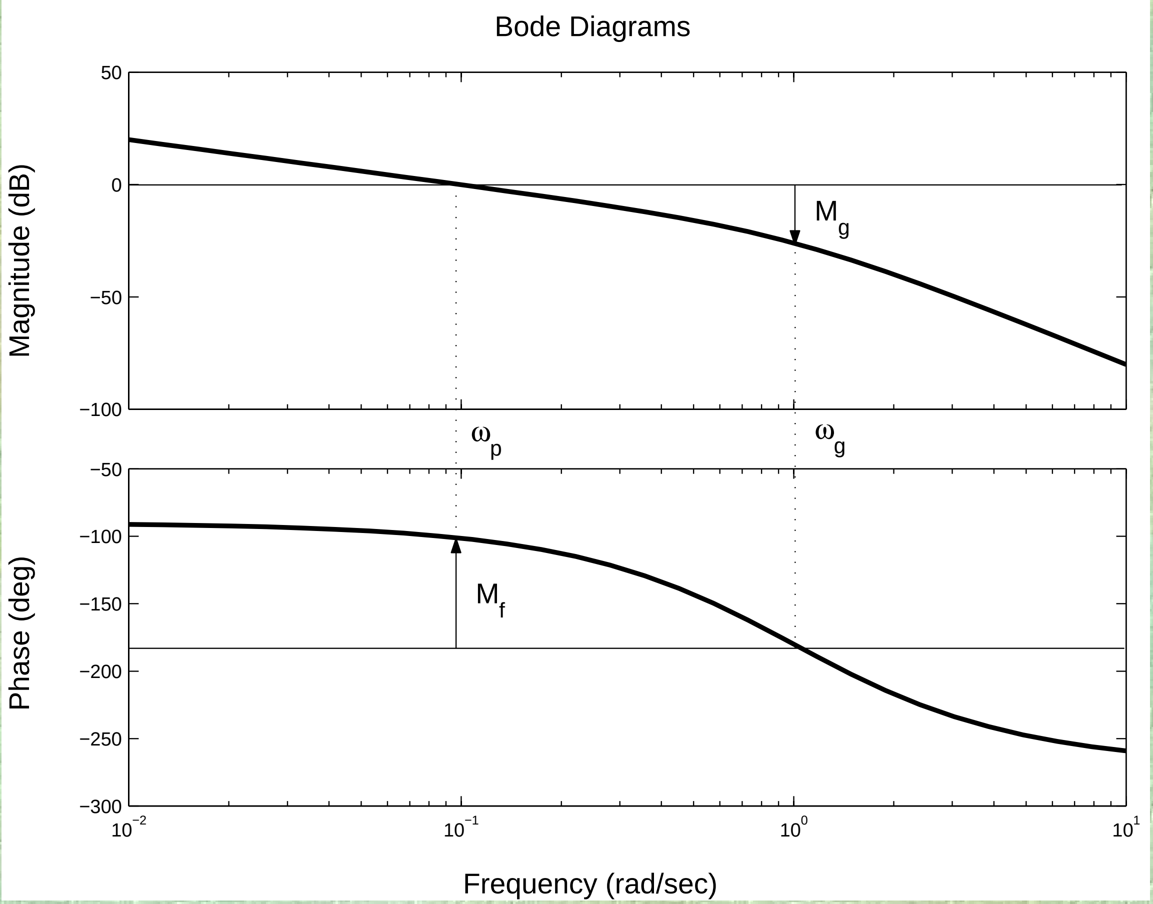
$$M_g \triangleq -20\log_{10}(|a|)$$

$$M_f \triangleq \phi$$

(b) Peak sensitivity:

Since $S_0 = \dfrac{1}{1+G_0C}$, then S_0 is a maximum at the frequency where $G_0(jw)C(jw)$ is closest to the point -1. The peak sensitivity is thus $^1/_\eta$ - (see Figure 5.7).

Figure 5.8: *Stability margins in Bode diagrams*



Robustness

So far, we have only considered the effect that the controller has on the nominal closed loop formed with the nominal model for the plant. However, in practice, we are usually interested, not only in this nominal performance, but also the true performance achieved when the controller is applied to the true plant. This is the so called "Robustness" issue. We will show below that the nominal sensitivities do indeed tell us something about the true or achieved sensitivities.

Achieved Sensitivities

We contrast the nominal sensitivities derived previously with the achieved (or true) sensitivities when the controller $C(s)$ is applied to the calibration model, $G(s)$. This leads to the following calibration sensitivities:

$$
\begin{aligned}
T(s) &\triangleq \frac{G(s)C(s)}{1+G(s)C(s)} = \frac{B(s)P(s)}{A(s)L(s)+B(s)P(s)} \\
S(s) &\triangleq \frac{1}{1+G(s)C(s)} = \frac{A(s)L(s)}{A(s)L(s)+B(s)P(s)} \\
S_i(s) &\triangleq \frac{G(s)}{1+G(s)C(s)} = \frac{B(s)L(s)}{A(s)L(s)+B(s)P(s)} \\
S_u(s) &\triangleq \frac{C(s)}{1+G(s)C(s)} = \frac{A(s)P(s)}{A(s)L(s)+B(s)P(s)}
\end{aligned}
$$

Relationship to Modelling Errors

The achieved sensitivity functions are given in terms of the nominal sensitivities as follows:

$$\begin{aligned} S(s) &= S_o(s)S_\Delta(s) \\ T(s) &= T_o(s)(1+G_\Delta(s))S_\Delta(s) \\ S_i(s) &= S_{io}(s)(1+G_\Delta(s))S_\Delta(s) \\ S_u(s) &= S_{uo}(s)S_\Delta(s) \\ S_\Delta(s) &= \frac{1}{1+T_o(s)G_\Delta(s)} \end{aligned}$$

Where $G_\Delta(s)$ is the multiplicative modelling error.

Robust Stability

We are concerned with the case where the nominal model and the true plant differ. It is then necessary that, in addition to nominal stability, we check that stability is retained when the true plant is controlled by the same controller. We call this property *robust stability*.

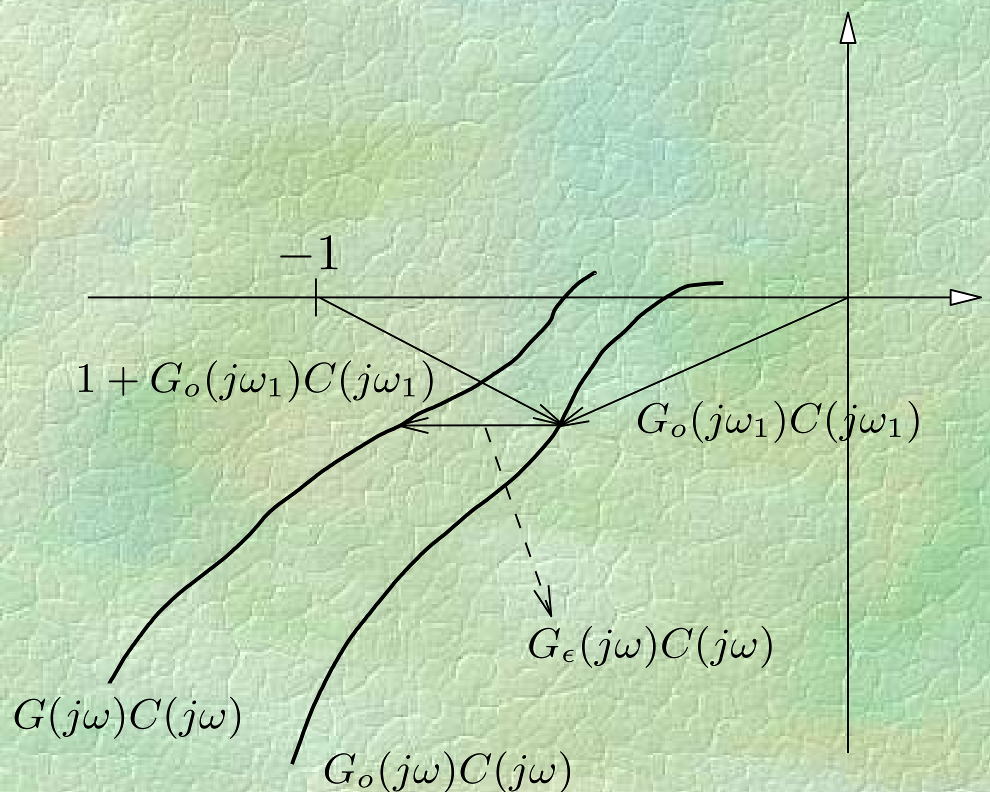
Theorem 5.3 (Robust stability theorem):

Consider a plant with nominal transfer function $G_0(s)$ and true transfer function given by $G(s)$. Assume that $C(s)$ is the transfer function of a controller which achieves nominal internal stability. Also assume that $G(s)C(s)$ and $G_0(s)C(s)$ have the same number of unstable poles. Then a sufficient condition for stability of the true feedback loop obtained by applying the controller to the true plant is that ...

$$|T_o(j\omega)||G_\Delta(j\omega)| = \left|\frac{G_o(j\omega)C(j\omega)}{1+G_o(j\omega)C(j\omega)}\right||G_\Delta(j\omega)| < 1 \qquad \forall\omega$$

where $G_\Delta(jw)$ is the frequency response of the multiplicative modeling error (MME).

Proof: *Consider the Nyquist plot for the nominal and the true loop*



From that figure we see that the same number of encirclements occur if

$$|G_\epsilon(j\omega)C(j\omega)| < |1 + G_o(j\omega)C(j\omega)| \qquad \forall\omega$$

this is equivalent to

$$\frac{|G_\Delta(j\omega)G_o(j\omega)C(j\omega)|}{|1 + G_o(j\omega)C(j\omega)|} < 1$$

∇∇∇

Example

In a feedback control loop, the open loop transfer function is given by

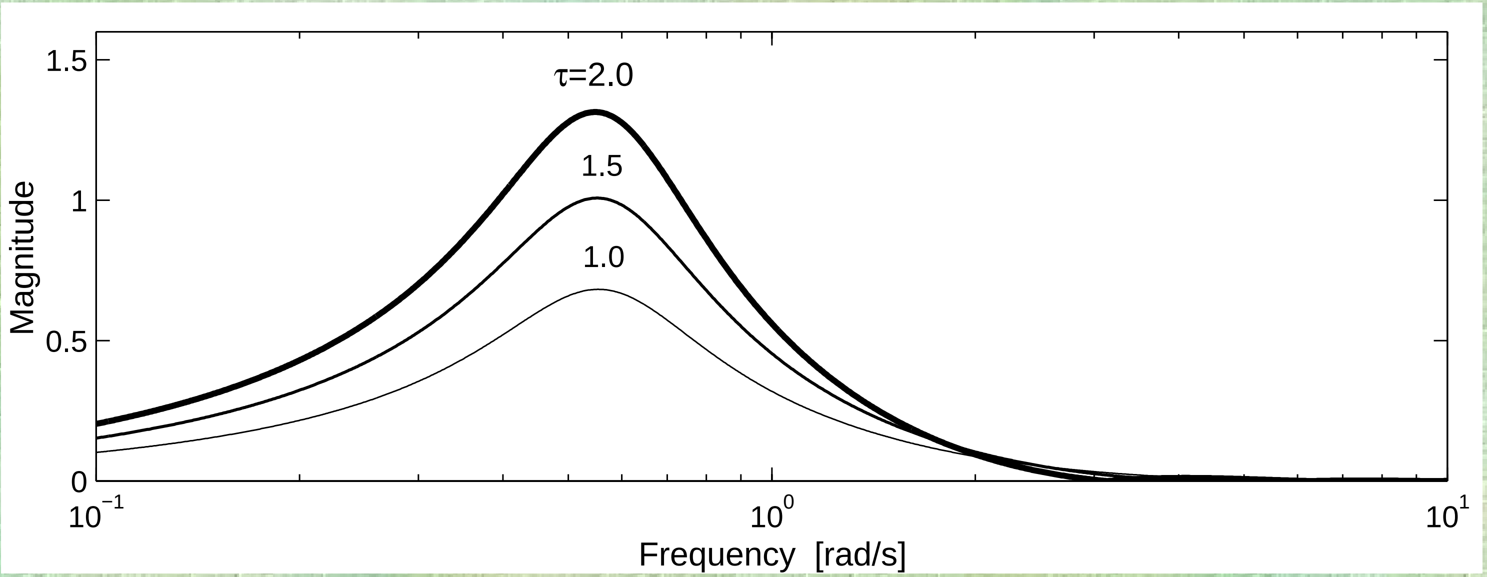
$$G_o(s)C(s) = \frac{0.5}{s(s+1)^2}$$

and the true plant transfer function is

$$G(s) = e^{-s\tau} G_o(s)$$

Use the Robust Stability Theorem to obtain a bound on the (unmodelled delay) τ which guarantees closed loop stability.

Figure 5.10: *Magnitude of the frequency response of $T_0(s)G_\Delta(s)$ for different values of τ*



Note that $|T_0(jw)G_\Delta(jw)| < 1, \ \forall w$ for $\tau \leq 1.5$.

Summary

- This chapter introduced the fundamentals of SISO feedback control loop analysis.

- Feedback introduces a cyclical dependence between controller and system:
  - the controller action affects the systems outputs,
  - and the system outputs affect the controller action.

- Well designed, feedback can
  - make an unstable system stable;
  - increase the response speed;
  - decrease the effects of disturbances;
  - decrease the effects of system parameter uncertainties, and more.

- Poorly designed, feedback can
  - introduce instabilities into a previously stable system;
  - add oscillations into a previously smooth response;
  - result in high sensitivity to measurement noise;
  - result in sensitivity to structural modeling errors, and more.