

Chapter 22

Design via Optimal Control Techniques

In the author's experience, industrial control system design problems can be divided into four categories:

1. Relatively simple loops for which PID design gives a very satisfactory solution (*see Chapters 6 and 7*).
2. Slightly more complex problems where an additional feature beyond PID yields significant performance advantages. Two key tools that can be used to considerably advantage in this context are feedforward control (*Chapter 10*) and the Smith Predictor for plants with significant time delays (*Chapters 7 and 15*).

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3. Systems involving significant interactions but where some form of preliminary compensation essential converts the problem into separate non-interacting loops which then fall under categories 1 and 2 above (*Chapter 21*).
 4. Difficult problems which require some form of computer assisted optimization for their solution. (*This is the topic of the current chapter and Chapter 23*).

As a rough guideline: 95% of control problems fall into category 1 above; 4% fall into category 2 or 3. The remaining 1% fall into category 4.

However, the relative low frequency of occurrence of the problems in category 4 is not representative of their importance. Indeed, it is often this 1% of *hard* problems where the real benefits of control system design can be achieved. They are often the *make or break* problems.

We will emphasize methods for solving these *tougher* problems based on optimal control theory. There are three reasons for this choice:

1. It is relatively easy to understand
2. It has been used in a myriad of applications. (*Indeed, the authors have used these methods on approximately 20 industrial applications*).
3. It is a valuable precursor to other advanced methods - e.g., Model Predictive Control, which is explained in the next chapter.

The analysis presented in this chapter builds on the results in Chapter 18, where state space design methods were briefly described in the SISO context. We recall, from that chapter, that the two key elements were

- ◆ *state estimation by an observer*
- ◆ *state-estimate feedback*

State-Estimate Feedback

Consider the following MIMO state space model having m inputs and p outputs.

$$\dot{x}(t) = \mathbf{A}_o x(t) + \mathbf{B}_o u(t)$$

$$y(t) = \mathbf{C}_o x(t)$$

By analogy with state-estimate feedback in the SISO case (*as in Chapter 7*), we seek a matrix $\mathbf{K} \in \mathbb{R}^{m \times n}$ and a matrix $\mathbf{J} \in \mathbb{R}^{n \times p}$ such that $(\mathbf{A}_o - \mathbf{B}_o \mathbf{K})$ and $(\mathbf{A}_o - \mathbf{J} \mathbf{C}_o)$ have their eigenvalues in the LHP. Further we will typically require that the closed-loop poles reside in some specified region in the left-half plane. Tools such as MATLAB provide solutions to these problems.

Example 22.1

Consider a MIMO plant having the nominal model

$$\mathbf{G}_o(s) = \frac{1}{s(s+1)(s+2)} \begin{bmatrix} 2(s+1) & -0.5s(s+1) \\ s & 2s \end{bmatrix}$$

Say that the plant has step-type input disturbances in both channels.

Using state-estimate feedback ideas, design a multivariable controller which stabilizes the plant and, at the same time, ensures zero steady-state error for constant references and disturbances.

We first build state space models $(\mathbf{A}_p, \mathbf{B}_p, \mathbf{C}_p, \mathbf{0})$ and $(\mathbf{A}_d, \mathbf{B}_d, \mathbf{C}_d, \mathbf{0})$ for the plant and for the input disturbances, respectively.

We estimate not only the plant state $x_p(t)$ but also the disturbance vector $d_i(t)$. We then form the control law

$$u(t) = -K_p \hat{x}(t) - \hat{d}_i(t) + \bar{r}(t)$$

One pair of possible state space models is

$$\dot{x}_p(t) = \mathbf{A}_p x_p(t) + \mathbf{B}_p u(t)$$

$$y(t) = \mathbf{C}_p x_p(t)$$

$$\dot{x}_d(t) = \mathbf{A}_d x_d(t) + \mathbf{B}_d u(t)$$

$$d_i(t) = \mathbf{C}_d x_d(t)$$

where

$$\mathbf{A}_p = \begin{bmatrix} -3 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{B}_p = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & -0.5 \\ 1 & 0 \end{bmatrix}; \quad \mathbf{C}_p = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{A}_d = \mathbf{0};$$

$$\mathbf{B}_d = \mathbf{0};$$

$$\mathbf{C}_d = \mathbf{I}_2$$

The augmented state space model, $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{0})$ is then given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_p & \mathbf{B}_p \mathbf{C}_d \\ \mathbf{0} & \mathbf{A}_d \end{bmatrix} = \begin{bmatrix} \mathbf{A}_p & \mathbf{B}_p \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_p \\ \mathbf{B}_d \end{bmatrix} = \begin{bmatrix} \mathbf{B}_p \\ \mathbf{0} \end{bmatrix} \quad \mathbf{C} = [\mathbf{C}_p \quad \mathbf{0}]$$

leading to a model with six states.

We then compute the observer gain \mathbf{J} , choosing the six observer poles located at $-5, -6, -7, -8, -9, -10$. This is done using the MATLAB command `place` for the pair $(\mathbf{A}^T, \mathbf{C}^T)$.

Next we compute the feedback gain \mathbf{K} . We note that it is equivalent (with $\bar{r}(t) = 0$) to

$$u(t) = - [\mathbf{K}_p \quad \mathbf{C}_d] \begin{bmatrix} \hat{x}_p(t) \\ \hat{x}_d(t) \end{bmatrix} \implies \mathbf{K} = [\mathbf{K}_p \quad \mathbf{I}_2]$$

i.e., we need only compute \mathbf{K}_p . This is done by using the MATLAB command `place` for the pair $(\mathbf{A}_p, \mathbf{B}_p)$. The poles in this case are chosen at $-1.5 \pm j1.32, -3$ and -5 .

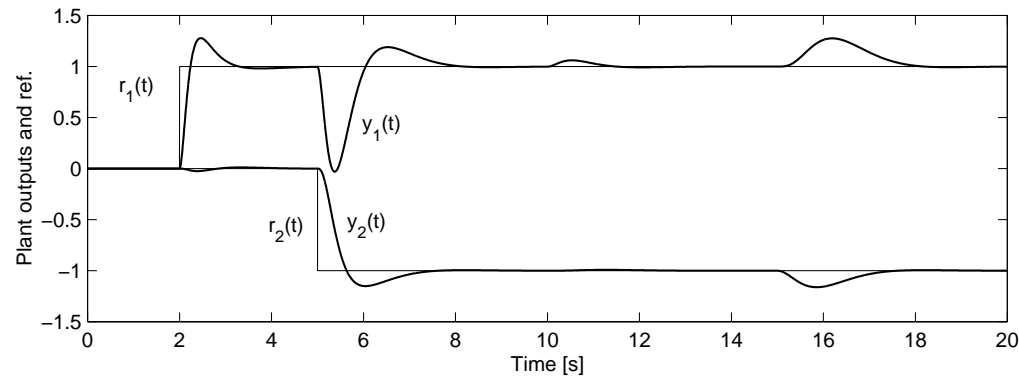
The design is evaluated by applying step references and input disturbances in both channels, as follows:

$$r_1(t) = \mu(t - 2); \quad r_2(t) = -\mu(t - 5); \quad d_i^{(1)}(t) = \mu(t - 10); \quad d_i^{(2)}(t) = \mu(t - 15)$$

where $d_i^{(1)}(t)$ and $d_i^{(2)}(t)$ are the first and second components of the input-disturbance vector respectively.

The results are shown on the next slide.

Figure 22.1: *MIMO design based in state-estimate feedback*



The above results indicate that the design is quite satisfactory. Note that there is strong coupling but decoupling was not part of the design specification.

We next turn to an alternative procedure that deals with the MIMO case via optimization methods. A particularly nice approach for the design of \mathbf{K} and \mathbf{J} is to use quadratic optimization because it leads to simple closed-form solutions.

Dynamic Programming and Optimal Control

We begin at a relatively abstract nonlinear level but our ultimate aim is to apply these ideas to the linear case.

The Optimal Control Problem

Consider a general nonlinear system with input $u(t) \in \mathbb{R}^m$, described in state space form by

$$\frac{dx(t)}{dt} = f(x(t), u(t), t)$$

Problem: (*General optimal control problem*). Find an optimal input $u_o(t)$, for $t \in [t_o, t_f]$, such that

$$u^o(t) = \arg \min_{u(t)} \left\{ \int_{t_o}^{t_f} \mathcal{V}(x, u, t) dt + g(x(t_f)) \right\}$$

where $\mathcal{V}(s, u, t)$ and $g(x(t_f))$ are nonnegative functions.

Necessary Condition for Optimality

Theorem 22.1: (*Optimality Principle Bellman*). If $\{u(t) = u^o(t), t \in [t_o, t_f]\}$ is the optimal solution for the above problem, then $u^o(t)$ is also the optimal solution over the (sub)interval $[t_o + \Delta t, t_f]$, where $t_o < t_o + \Delta t < t_f$.

Proof: See the book. The essential idea is that any part of an optimal trajectory is necessarily optimal in its own right.

We will next use the above theorem to derive necessary conditions for the optimal u . The idea is to consider a general time interval $[t, t_f]$, where $t \in [t_o, t_f]$, and then to use the Optimality Principle with an infinitesimal time interval $[t, t + \Delta t]$.

Some straightforward analysis leads to the following equations for the optimal cost:

$$-\frac{\partial J^o(x(t), t)}{\partial t} = \mathcal{V}(x(t), \mathcal{U}, t) + \left[\frac{\partial J^o(x(t), t)}{\partial x} \right]^T f(x(t), \mathcal{U}, t)$$

The solution for this equation must satisfy the boundary condition

$$J^o(x(t_f), t_f) = g(x(t_f))$$

At this stage we cannot proceed further without being more specific about the nature of the original problem. We also note that we have implicitly assumed that the function $J^o(x(t), t)$ is well behaved, which means that it is continuous in its arguments and that it can be expanded in a Taylor series.

The Linear Quadratic Regulator (LQR)

We next apply the above general theory to the following problem.

Problem: (*The LQR problem*). Consider a linear time-invariant system having a state space model, as defined below:

$$\begin{aligned}\frac{dx(t)}{dt} &= \mathbf{A}x(t) + \mathbf{B}u(t) & x(t_o) &= x_o \\ y(t) &= \mathbf{C}x(t) + \mathbf{D}u(t)\end{aligned}$$

We aim to drive the initial state x_o to the smallest possible value as soon as possible in the interval $[t_o, t_f]$, but without spending too much control effort.

In particular, we aim to optimize

$$J_u(x(t_o), t_o) = \int_{t_o}^{t_f} [x(t)^T \Psi x(t) + u(t)^T \Phi u(t)] dt + x(t_f)^T \Psi_f x(t)$$

where $\Psi \in \mathbb{R}^{n \times n}$ and $\Psi_f \in \mathbb{R}^{n \times n}$ are symmetric nonnegative definite matrices and $\Phi \in \mathbb{R}^{m \times m}$ is a symmetric positive definite matrix.

Note that this is a special case of the general cost function given early - this one is quadratic in the states and controls. Hence the name *Linear Quadratic Optimal Control*.

To solve this problem, the theory summarized above can be used. We first make the following connections between the general optimal problem and the LQR problem:

$$f(x(t), u(t), t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$

$$\mathcal{V}(x, u, t) = x(t)^T \mathbf{\Psi}x(t) + u(t)^T \mathbf{\Phi}u(t)$$

$$g(x(t_f)) = x(t_f)^T \mathbf{\Psi}_f x(t_f)$$

Simple application of the general conditions for optimality leads to

$$u^o(t) = -\mathbf{K}_u(t)x(t)$$

where $\mathbf{K}_u(t)$ is a time varying gain, given by

$$\mathbf{K}_u(t) = \mathbf{\Phi}^{-1}\mathbf{B}^T\mathbf{P}(t)$$

where $\mathbf{P}(t)$ satisfies

$$-\frac{d\mathbf{P}(t)}{dt} = \mathbf{\Psi} - \mathbf{P}(t)\mathbf{B}\mathbf{\Phi}^{-1}\mathbf{B}^T\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A} + \mathbf{A}^T\mathbf{P}(t)$$

The above equation is known as the Continuous Time Dynamic Riccati Equation (CTDRE). This equation has to be solved backwards in time, to satisfy the boundary condition:

$$\mathbf{P}(t_f) = \mathbf{\Psi}_f$$

Some brief history of this equation is contained in the excellent book:

Bittanti, Laub, Williams, “The Riccati Equation” ,
Springer Verlag, 1991.

Some extracts are given below.

Some History of the Riccati Equation

“Towards the turn of the seventeenth century, when the baroque was giving way to the enlightenment, there lived in the Republic of Venice a gentleman, the father of nine children, by the name of Jacopo Francesco Riccati. On the cold New Year’s Eve of 1720, he wrote a letter to his friend Giovanni Rizzetti, where he proposed two new differential equations. In modern symbols, these equations can be written as follows.

$$\dot{x} = \alpha x^2 + \beta t^m$$
$$\dot{x} = \alpha x^2 + \beta t + \lambda t^2$$

Where m is a constant. This is probably the first document witnessing the early days of the Riccati Equation, an equation which was to become of paramount importance in the centuries to come.”

Who was Riccati ?

“Count Jacopo Riccati was born in Venice on May 28, 1676. His father, a nobleman, died when he was only ten years old. The boy was raised by his mother, who did not marry again, and by a paternal uncle, who recognized unusual abilities in his nephew and persuaded Jacopo Francesco’s mother to have him enter a Jesuit college in Brescia. Young Riccati enrolled at this college in 1687, probably with no intention of ever becoming a scientist. Indeed, at the end of his studies at the college, in 1693, he enrolled at the university of Padua as a student of law. However, following his natural inclination, he also attended classes in astronomy given by Father Stefano degli Angeli, a former pupil of Bonaventura Cavalieri. Father Stefano was fond of Isaac Newton’s *Philosophiae Naturalis Principia*, which he passed onto young Riccati around 1695. This is probably the event which caused Riccati to turn from law to science.”

“After graduating on June 7, 1696, he married Elisabetta dei Conti d’Onigo on October 15, 1696. She bore him 18 children, of whom 9 survived childhood. Amongst them, Vincenzo (b.1707, d.1775), a mathematical physicist, and Giordano (b.1709, d.1790) a scholar with many talents but with a special interest for architecture and music, are worth mentioning.

Riccati spent most of his life in Castelfranco Veneto, a little town located in the beautiful country region surrounding Venice. Besides taking care of his family and his large estate, he was in charge of the administration of Castelfranco Veneto, as Provveditore (Mayor) of that town, for nine years during the period 1698-1729. He also owned a house in the nearby town of Treviso, where he moved after the death of his wife (1749), and where his children had been used to spending a good part of each year after 1747”.

Count Jacopo Francesco Riccati



Returning to the theory of Linear Quadratic Optimal Control, we note that the theory holds equally well for time-varying systems - i .e., when \mathbf{A} , \mathbf{B} , Φ , Ψ are all functions of time. This follows since no explicit (or implicit) use of the time invariance of these matrices was used in the derivation. However, in the time-invariant case, one can say much more about the properties of the solution. This is the subject of the next section.

Properties of the Linear Quadratic Optimal Regulator

Here we assume that A , B , Φ , Ψ are all time-invariant. We will be particularly interested in what happens at $t \rightarrow \infty$. We will summarize the key results here.

Quick Review of Properties

We make the following simplifying assumptions:

- (i) *The system (\mathbf{A}, \mathbf{B}) is stabilizable from $u(t)$.*
- (ii) *The system states are all adequately seen by the cost function. Technically, this is stated as requiring that $(\Psi^{1/2}, \mathbf{A})$ be detectable.*

Under these conditions, the solution to the CTDRE, $\mathbf{P}(t)$, converges to a steady-state limit \mathbf{P}^s_∞ as $t_f \rightarrow \infty$. This limit has two key properties:

- ❖ \mathbf{P}^s_∞ is the only nonnegative solution of the matrix algebraic Riccati equation $\mathbf{0} = \mathbf{\Psi} - \mathbf{P}_\infty \mathbf{B} \mathbf{\Phi}^{-1} \mathbf{B}^T \mathbf{P}_\infty + \mathbf{P}_\infty \mathbf{A} + \mathbf{A}^T \mathbf{P}_\infty$ obtained by setting $d\mathbf{P}(t)/dt = 0$ in

$$-\frac{d\mathbf{P}(t)}{dt} = \mathbf{\Psi} - \mathbf{P}(t) \mathbf{B} \mathbf{\Phi}^{-1} \mathbf{B}^T \mathbf{P}(t) + \mathbf{P}(t) \mathbf{A} + \mathbf{A}^T \mathbf{P}(t)$$

- ❖ When this steady-state value is used to generate a feedback control law, then the resulting closed-loop system is *stable*.

More Detailed Review of Properties

Lemma 22.1: If $\mathbf{P}(t)$ converges as $t_f \rightarrow \infty$, then the limiting value \mathbf{P}_∞ satisfies the following Continuous-Time Algebraic Riccati Equation (CTARE):

$$0 = \Psi - \mathbf{P}_\infty \mathbf{B} \Phi^{-1} \mathbf{B}^T \mathbf{P}_\infty + \mathbf{P}_\infty \mathbf{A} + \mathbf{A}^T \mathbf{P}_\infty$$

The above algebraic equation can have many solutions. However, provided (\mathbf{A}, \mathbf{B}) is stabilizable and $(\mathbf{A}, \Psi^{1/2})$ has no unobservable modes on the imaginary axis, then there exists a unique *positive semidefinite solution* \mathbf{P}_∞^s to the CTARE having the property that the system matrix of the closed-loop system, $\mathbf{A} - \Phi^{-1} \mathbf{B}^T \mathbf{P}_\infty^s$, has all its eigenvalues in the OLHP. We call this particular solution the *stabilizing solution* of the CTARE. Other properties of the stabilizing solution are as follows:

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-
- (a) If $(\mathbf{A}, \Psi^{1/2})$ is detectable, the stabilizing solution is the only nonnegative solution of the CTARE.
- (b) If $(\mathbf{A}, \Psi^{1/2})$ has unobservable modes in the OLHP, then the stabilizing solution is not positive definite.
- (c) If $(\mathbf{A}, \Psi^{1/2})$ has an unobservable pole outside the OLHP, then, in addition to the stabilizing solution, there exists at least one other nonnegative solution to the CTARE. However, in this case, the stabilizing solution satisfies $\mathbf{P}_\infty^s - \bar{\mathbf{P}}_\infty \geq 0$, where $\bar{\mathbf{P}}_\infty$ is any other solution of the CTARE.

Proof: See the book.

Thus we see that the *stabilizing solution* of the CTRAE has the key property that, when this is used to define a state variable feedback gain, then the resulting closed loop system is guaranteed stable.

We next study the convergence of the solutions of the CTRDE (*a differential equation*) to particular solutions of the CTRAE (*an algebraic equation*).

We will be particularly interested in those conditions which guarantee convergence to the stabilizing solution.

Convergence of the solution of the CTDRE to the stabilizing solution of the CTARE is addressed in the following lemma.

Lemma 2.22: Provided that (\mathbf{A}, \mathbf{B}) is stabilizable, that $(\mathbf{A}, \Psi^{1/2})$ has no unobservable poles on the imaginary axis, and that the terminal condition satisfies: $\Psi_f > \mathbf{P}_\infty^s$, then

$$\lim_{t_f \rightarrow \infty} \mathbf{P}(t) = \mathbf{P}_\infty^s$$

(If we strengthen the condition of Ψ to require that $(\mathbf{A}, \Psi^{1/2})$ is detectable, then $\Psi_f \geq 0$ suffices).

Proof: See the book.

Example

Consider the scalar system

$$\dot{x}(t) = ax(t) + u(t)$$

and the cost function

$$J = \psi_f x(t_f)^2 + \int_0^{t_f} (\psi x(t)^2 + u(t)^2) dt$$

The associated CTDRE is

$$\dot{P}(t) = -2aP(t) + P(t)^2 - \psi; \quad P(t_f) = \psi_f$$

and the CTARE is

$$(P_\infty^s)^2 - 2aP_\infty^s - \psi = 0$$

Case 1: $\psi \neq 0$

Here, $(\mathbf{A}, \Psi^{1/2})$ is completely observable (*and thus detectable*). There is only one nonnegative solution of the CTARE. This solution coincides with the stabilizing solution. Making the calculations, we find that the only nonnegative solution of the CTARE is

$$P_{\infty}^s = \frac{2a + \sqrt{4a^2 + 4\psi}}{2}$$

leading to the following gain:

$$K_{\infty}^s = a + \sqrt{a^2 + \psi}$$

The corresponding closed-loop pole is at

$$p_{cl} = -\sqrt{a^2 + \psi}$$

This is clearly in the LHP, verifying that the solution is indeed the stabilizing solution.

Other cases are considered in the book.

To study the convergence of the solutions, we again consider:

Case 1: $\psi \neq 0$

Here $(\mathbf{A}, \Psi^{1/2})$ is completely observable. Then $\mathbf{P}(t)$ converges to \mathbf{P}^s_∞ for any $\psi_f \geq 0$.

Linear quadratic regulator theory is a powerful tool in control-system design. We illustrate its versatility in the next section by using it to solve the so-called Model Matching Problem (MMP).

Model Matching Based on Linear Quadratic Optimal Regulators

Many problems in control synthesis can be reduced to a problem of the following type:

Given two stable transfer functions $\mathbf{M}(s)$ and $\mathbf{N}(s)$, find a stable transfer function $\Gamma(s)$ so that $\mathbf{N}(s)\Gamma(s)$ is close to $\mathbf{M}(s)$ in a quadratic norm sense.

When $\mathbf{M}(s)$ and $\mathbf{N}(s)$ are matrix transfer functions, we need to define a suitable norm to measure *closeness*. By way of illustration, we consider a matrix $\mathbf{A} = [a_{ij}] \in \mathbb{C}^{p \times m}$ for which we define the *Fröbenius norm* as follows

$$\|\mathbf{A}\|_F = \sqrt{\text{trace} \mathbf{A}^H \mathbf{A}} = \sqrt{\sum_{i=1}^m \sum_{j=1}^p |a_{ij}|^2}$$

Using this norm, a suitable synthesis criterion for the Model Matching Problem described earlier might be:

$$\Gamma^\circ = \arg \min_{\Gamma \in \mathcal{S}} J_\Gamma$$

where

$$J_\Gamma = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathbf{M}(j\omega) - \mathbf{N}(j\omega)\Gamma(j\omega)\|_F^2 d\omega$$

and where \mathcal{S} is the class of stable transfer functions.

This problem can be converted into vector form by vectorizing \mathbf{M} and Γ . For example, say that Γ is constrained to be lower triangular and that \mathbf{M} , \mathbf{N} , and Γ are 3×3 , 3×2 , and 2×2 matrices, respectively; then we can write

$$J_{\Theta} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathbf{V}(j\omega) - \mathbf{W}(j\omega)\Theta(j\omega)\|_2^2 d\omega$$

where $\|\cdot\|_2$ denotes the usual Euclidean vector norm and where, in this special case,

$$\mathbf{V}(s) = \begin{bmatrix} M_{11}(s) \\ M_{12}(s) \\ M_{21}(s) \\ M_{22}(s) \\ M_{31}(s) \\ M_{32}(s) \end{bmatrix}; \quad \mathbf{W}(s) = \begin{bmatrix} N_{11}(s) & N_{12}(s) & 0 \\ 0 & 0 & N_{12}(s) \\ N_{21}(s) & N_{22}(s) & 0 \\ 0 & 0 & N_{22}(s) \\ N_{31}(s) & N_{32}(s) & 0 \\ 0 & 0 & N_{32}(s) \end{bmatrix}; \quad \mathbf{\Theta}(s) = \begin{bmatrix} \Gamma_{11}(s) \\ \Gamma_{21}(s) \\ \Gamma_{22}(s) \end{bmatrix}$$

Conversion to Time Domain

We next select a state space model for $\mathbf{V}(s)$ and $\mathbf{W}(s)$ of the form

$$\mathbf{V}(s) = \mathbf{C}_1[s\mathbf{I} - \mathbf{A}_1]^{-1}\mathbf{B}_1$$
$$\mathbf{W}(s) = \mathbf{C}_2[s\mathbf{I} - \mathbf{A}_2]^{-1}\mathbf{B}_2$$

Before proceeding to solve the model-matching problem, we make a slight generalization. In particular, it is sometimes desirable to restrict the size of Θ . We do this by generalizing the cost function by introducing an extra term that weights Θ . This leads to

$$J_{\Theta} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \|\mathbf{V}(j\omega) - \mathbf{W}(j\omega)\Theta(j\omega)\|_{\Gamma}^2 + \|\Theta(j\omega)\|_{\mathbf{R}}^2 \right\} d\omega$$

where Γ and \mathbf{R} are nonnegative symmetrical matrices.

We can then apply Parseval's theorem to convert J_{Θ} into the time domain. The transfer functions are stable and strictly proper, so this yields

$$J_{\Theta} = \int_0^{\infty} \left\{ \|y_1(t) - y_2(t)\|_{\Gamma}^2 + \|u(t)\|_{\mathbf{R}}^2 \right\} dt$$

where

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix} u(t); & \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned}$$

In detail we have

$$J_{\Theta} = \int_0^{\infty} \{x(t)^T \mathbf{\Psi} x(t) + u(t)^T \mathbf{R} u(t)\} dt$$

where $x(t) = [x_1(t)^T \quad x_2(t)^T]$ and

$$\mathbf{\Psi} = \begin{bmatrix} \mathbf{C}_1^T \\ -\mathbf{C}_2^T \end{bmatrix} \mathbf{\Gamma} [\mathbf{C}_1 \quad -\mathbf{C}_2]$$

We recognize this as a standard LQR problem, where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix}$$

Note that, to achieve the transformation of the model-matching problem into a LQR problem, the key step is to link $L^{-1}[\Theta(s)]$ to $u(t)$.

Solution

We are interested in expressing $u(t)$ as a function of $x(t)$ - i.e.,

$$u(t) = -\mathbf{K}x(t) = -[\mathbf{K}_1 \quad \mathbf{K}_2] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

such that J_{Θ} is minimized. The optimal value of \mathbf{K} is given by the solution to the LQR problem. We will also assume that the values of \mathbf{A} , \mathbf{B} , Φ , etc. are such that \mathbf{K} corresponds to a stabilizing solution.

The final input $u(t)$ satisfies

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t) & x(0) &= [\mathbf{B}_1^T \quad \mathbf{0}]^T \\ u(t) &= -\mathbf{K}x(t) \end{aligned}$$

In transfer-function form, this is

$$U(s) = \Theta(s) = -\mathbf{K} (s\mathbf{I} - \mathbf{A} + \mathbf{BK})^{-1} \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}$$

which, upon our using the special structure of \mathbf{A} , \mathbf{B} , and \mathbf{K} , yields

$$\Theta(s) = [-\mathbf{I} + \mathbf{K}_2 (s\mathbf{I} - \mathbf{A}_2 + \mathbf{B}_2\mathbf{K}_2)^{-1} \mathbf{B}_2] \mathbf{K}_1 (s\mathbf{I} - \mathbf{A}_1)^{-1} \mathbf{B}_1$$

Discrete-Time Optimal Regulators

The theory for optimal quadratic regulators for continuous-time systems can be extended in a straightforward way to provide similar tools for discrete-time systems. We will briefly summarize the main results.

Consider a discrete-time system having the following state space description:

$$\begin{aligned}x[k + 1] &= \mathbf{A}_q x[k] + \mathbf{B}_q u[k] \\ y[k] &= \mathbf{C}_q x[k]\end{aligned}$$

and the cost function

$$J_u(x[k_o], k_o) = \sum_{k_o}^{k_f} \left(x[k]^T \mathbf{\Psi} x[k] + u[k]^T \mathbf{\Phi} u[k] \right) + x[k_f]^T \mathbf{\Psi}_f x[k_f]$$

The optimal quadratic regulator is given by

$$u^o[k] = -\mathbf{K}_u[k]x[k]$$

where $\mathbf{K}_u[k]$ is a time-varying gain, given by

$$\mathbf{K}_u[k] = (\Phi + \mathbf{B}^T \mathbf{P}[k] \mathbf{B})^{-1} \mathbf{B}^T \mathbf{P}[k] \mathbf{A}$$

where $\mathbf{P}[k]$ satisfies the following **Discrete Time Dynamic Riccati Equation (DTDRE)**.

$$\mathbf{P}[k] = \mathbf{A}^T \left(\mathbf{P}[k+1] - \mathbf{P}[k+1] \mathbf{B} (\Phi + \mathbf{B}^T \mathbf{P}[k+1] \mathbf{B})^{-1} \mathbf{B}^T \mathbf{P}[k+1] \right) \mathbf{A} + \Psi$$

This equation must also be solved backwards, subject to the boundary condition

$$\mathbf{P}[k_f] = \Psi_f$$

The steady-state ($k_f \rightarrow \infty$) version of the control law is given by

$$u^o[k] = -\mathbf{K}_\infty x[k] \quad \text{where} \quad \mathbf{K}_\infty = (\Phi + \mathbf{B}^T \mathbf{P}_\infty \mathbf{B})^{-1} \mathbf{B}^T \mathbf{P}_\infty \mathbf{A}$$

where \mathbf{K}_∞ and \mathbf{P}_∞ satisfy the associated **Discrete Time Algebraic Riccati Equation (DTARE)**:

$$\mathbf{A}^T \left(\mathbf{P}_\infty - \mathbf{P}_\infty \mathbf{B} (\Phi + \mathbf{B}^T \mathbf{P}_\infty \mathbf{B})^{-1} \mathbf{B}^T \mathbf{P}_\infty \right) \mathbf{A} + \Psi - \mathbf{P}_\infty = 0$$

with the property that $\mathbf{A} - \mathbf{B}\mathbf{K}_\infty$ has all its eigenvalues inside the stability boundary, provided that (\mathbf{A}, \mathbf{B}) is stabilizable and $(\mathbf{A}, \Psi^{1/2})$ has no unobservable modes on the unit circle.

Connections to Pole Assignment

Note that, under reasonable conditions, the steady-state LQR ensures closed-loop stability. However, the connection to the precise closed-loop dynamics is rather indirect; it depends on the choice of Ψ and Φ . Thus, in practice, one usually needs to perform some trial-and-error procedure to obtain satisfactory closed-loop dynamics.

In some circumstances, it is possible to specify a region in which the closed-loop poles should reside and to enforce this in the solution. A simple example of this is when we require that the closed-loop poles have real part to the left of $s = -\alpha$, for $\alpha \in \mathbb{R}^+$. This can be achieved by first shifting the axis by the transformation

$$v = s + \alpha$$

Then $\Re(s) = -\alpha \Rightarrow \Re\{v\} = 0$.

A slightly more interesting demand is to require that the closed-loop poles lie inside a circle with radius ρ and with center at $(-\alpha, 0)$, with $\alpha > \rho \geq 0$ - i.e., the circle is entirely within the LHP.

This can be achieved by using a two-step procedure:

-
- (i) We first transform the Laplace variable s to a new variable, ζ , defined as follows:

$$\zeta = \frac{s + \alpha}{\rho}$$

This takes the original circle in \boxed{s} to a unit circle in $\boxed{\zeta}$. The corresponding transformed state space model has the form

$$\zeta X(\zeta) = \frac{1}{\rho}(\alpha \mathbf{I} + \mathbf{A}_o)X(\zeta) + \frac{1}{\rho}\mathbf{B}_o U(\zeta)$$

-
- (ii) One then treats the above model as the state space description of a discrete-time system. So, solving the corresponding discrete optimal control problem leads to a feedback gain \mathbf{K} such that $1/\rho (\alpha \mathbf{I} + \mathbf{A}_0 - \mathbf{B}_0 \mathbf{K})$ has all its eigenvalues inside the unit disk. This in turn implies that, when the same control law is applied in continuous time, then the closed-loop poles reside in the original circle in \boxed{s} .

Example

Consider a 2×2 multivariable system having the state space model

$$\mathbf{A}_o = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 3 & -2 & 2 \end{bmatrix}; \quad \mathbf{B}_o = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & -1 \end{bmatrix}; \quad \mathbf{C}_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad \mathbf{D}_o = \mathbf{0}$$

Find a state-feedback gain matrix \mathbf{K} such that the closed-loop poles are all located in the disk with center at $(-\alpha; 0)$ and with radius ρ , where $\alpha = 6$ and $\rho = 2$.

We use the approach proposed above:

We first need the state space representation in the transformed space.

$$\mathbf{A} = \frac{1}{\rho}(\alpha\mathbf{I} + \mathbf{A}_o) \quad \text{and} \quad \mathbf{B} = \frac{1}{\rho}\mathbf{B}_o$$

The MATLAB command *dlqr*, with weighting matrices $\Psi = \mathbf{I}_3$ and $\Phi = \mathbf{I}_2$, is then used to obtain the optimal gain \mathbf{K}_ζ , which is

$$\mathbf{K} = \begin{bmatrix} 7.00 & -4.58 & 7.73 \\ 3.18 & 7.02 & -4.10 \end{bmatrix}$$

When this optimal gain is used in the original continuous-time system, the closed-loop poles, computed from $\det(s\mathbf{I} - \mathbf{A}_o + \mathbf{B}_o\mathbf{K}_\zeta) = 0$, are located at -5.13, -5.45, and -5.59. All these poles lie in the prescribed region, as expected.

Observer Design

Next, we turn to the problem of state estimation.

Here, we seek a matrix $\mathbf{J} \in \mathbb{R}^{n \times p}$ such that $\mathbf{A} - \mathbf{J}\mathbf{C}$ has its eigenvalues inside the stability region. Again, it is convenient to use quadratic optimization.

As a first step, we note that an observer can be designed for the pair (\mathbf{C}, \mathbf{A}) by simply considering an equivalent (*called dual*) control problem for the pair (\mathbf{A}, \mathbf{B}) . To illustrate how this is done, consider the *dual* system with

$$\mathbf{A}' = \mathbf{A}^T \qquad \mathbf{B}' = \mathbf{C}^T$$

Then, using any method for state-feedback design, we can find a matrix $\mathbf{K}' \in \mathbb{R}^{p \times n}$ such that $\mathbf{A}' - \mathbf{B}'\mathbf{K}'$ has its eigenvalues inside the stability region. Hence, if we choose $\mathbf{J} = (\mathbf{K}')^T$, then we have ensured that $\mathbf{A} - \mathbf{J}\mathbf{C}$ has its eigenvalues inside the stability region. Thus, we have completed the observer design.

The procedure leads to a stable state estimation of the form

$$\dot{\hat{x}}(t) = \mathbf{A}_o \hat{x}(t) + \mathbf{B}_o u(t) + \mathbf{J}(y(t) - \mathbf{C}\hat{x}(t))$$

Of course, using the tricks outlined above for state-variable feedback, one can also use transformation techniques to ensure that the poles describing the evolution of the observer error also end up in any region that can be related to either the continuous- or the discrete-time case by a rational transformation.

We will show how the above procedure can be formalized by using Optimal Filtering theory. The resulting optimal filter is called a Kalman filter.

Linear Optimal Filters

We will present one derivation of the optimal filters based on stochastic modeling of the noise. An alternative derivation based on model matching is given in the book.

Derivation Based on a Stochastic Noise Model

We show how optimal-filter design can be set -up as a quadratic optimization problem. This shows that the filter is *optimal* under certain assumptions regarding the signal-generating mechanism. In practice, this property is probably less important than the fact that the resultant filter has the right kind of *tuning knobs* so that it can be flexibly applied to a large range of problems of practical interest.

Details of the Stochastic Model

Consider a linear stochastic system of the form

$$dx(t) = \mathbf{A}x(t)dt + dw(t)$$

$$dy(t) = \mathbf{C}x(t)dt + dv(t)$$

where $dv(t)$ $dw(t)$ are known as *orthogonal increment processes*.

Since a formal treatment of stochastic differential equations is beyond the scope of this book, it suffices here to think of the formal notation $\dot{w}(t), \dot{v}(t)$ as white-noise processes with impulsive correlation:

$$E\{\dot{w}(t)\dot{w}(\zeta)^T\} = \mathbf{Q}\delta(t-\zeta)$$

$$E\{\dot{v}(t)\dot{v}(\zeta)^T\} = \mathbf{R}\delta(t-\zeta)$$

where $E\{\circ\}$ denotes mathematical expectation and $\delta(\circ)$ is the Dirac-delta function.

We can then *informally* write the model as

$$\begin{aligned}\frac{dx(t)}{dt} &= \mathbf{A}x(t) + \frac{dw(t)}{dt} \\ y'(t) = \frac{dy(t)}{dt} &= \mathbf{C}x(t) + \frac{dv(t)}{dt}\end{aligned}$$

For readers familiar with the notation of spectral density for random processes, we are simply requiring that the spectral density for $\dot{w}(t)$ and $\dot{v}(t)$ be \mathbf{Q} and \mathbf{R} , respectively.

Our objective will be to find a linear filter driven by $y'(t)$ that produces a state estimate $\hat{x}(t)$ having least possible error (*in a mean square sense*). We will optimize the filter by minimizing the quadratic function

$$J_t = \mathcal{E}\{\tilde{x}(t)\tilde{x}(t)^T\}$$

where

$$\tilde{x}(t) = \hat{x}(t) - x(t)$$

is the estimation error.

We will proceed to the solution of this problem in four steps.

Step 1:

Consider a time-varying version of the model given by

$$\frac{dx_z(t)}{dt} = \mathbf{A}_z(t)x(t) + \dot{w}_z(t)$$
$$y'_z(t) = \frac{dy_z(t)}{dt} = \mathbf{C}_z(t)x_z(t) + \dot{v}_z(t)$$

where $\dot{w}_z(t)$ and $\dot{v}_z(t)$ have zero mean and are uncorrelated, and

$$E\{\dot{w}_z(t)\dot{w}_z(\zeta)^T\} = \mathbf{Q}_z(t)\delta(t-\zeta)$$

$$E\{\dot{v}_z(t)\dot{v}_z(\zeta)^T\} = \mathbf{R}_z(t)\delta(t-\zeta)$$

For this model, we wish to compute

$$\bar{\mathbf{P}}(t) = E \{ x_z(t) x_z(t)^T \}.$$

We assume that $E \{ x_z(0) x_z(0)^T \} = \bar{\mathbf{P}}_0$, with $w_z(t)$ uncorrelated with the initial state $x_z(0) = x_{oz}$.

The solution to the model is easily seen to be

$$x_z(t) = \phi_z(t, 0)x_{oz} + \int_0^t \phi_z(t, \tau)\dot{w}_z(\tau)d\tau$$

where $\phi_z(t_2, t_1) \in \mathbb{R}^{n \times n}$ is the state transition matrix for the system. Then squaring and taking mathematical expectations, we have

$$\bar{\mathbf{P}}(t) = \mathcal{E}\{x_z(t)x_z(t)^T\} = \phi_z(t, 0)\bar{\mathbf{P}}_o\phi_z(t, 0)^T + \int_0^t \phi_z(t, \tau)\mathbf{Q}_z(\tau)\phi_z(t, \tau)^T d\tau$$

Differentiating the above equation and using the Leibnitz rule, we obtain

$$\frac{d\bar{\mathbf{P}}(t)}{dt} = \mathbf{A}_z \bar{\mathbf{P}}(t) + \bar{\mathbf{P}}(t) \mathbf{A}_z^T + \mathbf{Q}_z(t)$$

where we have also used the fact that $\frac{d}{dt} \phi(t, \tau) = \mathbf{A}_z(t) \phi(t, \tau)$.

Step 2:

We now return to the original problem: to obtain an estimate, $\hat{x}(t)$, for the state, $x(t)$. We make a simplifying assumption by fixing the *form* of the filter. That is, we assume the following linear form for the filter:

$$\frac{d\hat{x}(t)}{dt} = \mathbf{A}\hat{x}(t) + \mathbf{J}(t)[y'(t) - \mathbf{C}x(t)]$$

where $\mathbf{J}(t)$ is a time-varying gain yet to be determined.

Step 3:

Assume that we are also given an initial state estimate \hat{x}_o having the statistical property

$$\mathcal{E}\{(x(0) - \hat{x}_o)(x(0) - \hat{x}_o)^T\} = \mathbf{P}_o,$$

and assume, for the moment, that we are given some gain $\mathbf{J}(\tau)$ for $0 \leq \tau \leq t$. Derive an expression for

$$\begin{aligned}\mathbf{P}(t) &= \mathcal{E}\{(\hat{x}(t) - x(t))(\hat{x}(t) - x(t))^T\} \\ &= \mathcal{E}\{\tilde{x}(t)\tilde{x}(t)^T\}\end{aligned}$$

Solution: Subtracting the model from the filter format, we obtain

$$\frac{d\tilde{x}(t)}{dt} = (\mathbf{A} - \mathbf{J}(t)\mathbf{C})\tilde{x}(t) + \mathbf{J}(t)\dot{v}(t) - \dot{w}(t)$$

We see that this is a time-varying system, and we can therefore immediately apply the solution to Step 1, after making the following connections:

$$x_z(t) \rightarrow \tilde{x}(t); \quad \mathbf{A}_z(t) \rightarrow (\mathbf{A} - \mathbf{J}(t)\mathbf{C}); \quad \dot{w}_z(t) \rightarrow \mathbf{J}(t)\dot{v}(t) - \dot{w}(t)$$

to conclude

$$\frac{d\mathbf{P}(t)}{dt} = (\mathbf{A} - \mathbf{J}(t)\mathbf{C})\mathbf{P}(t) + \mathbf{P}(t)(\mathbf{A} - \mathbf{J}(t)\mathbf{C})^T + \mathbf{J}(t)\mathbf{R}\mathbf{J}(t)^T + \mathbf{Q}$$

subject to $\mathbf{P}(0) = \mathbf{P}_0$. Note that we have used the fact that $\mathbf{Q}_z(t) = \mathbf{J}(t)\mathbf{R}\mathbf{J}(t)^T + \mathbf{Q}$.

Step 4:

We next choose $\mathbf{J}(t)$, at each time instant, so that $\dot{\mathbf{P}}$ is as small as possible.

Solution: We complete the square on the right-hand side of

$$\frac{d\mathbf{P}(t)}{dt} = (\mathbf{A} - \mathbf{J}(t)\mathbf{C})\mathbf{P}(t) + \mathbf{P}(t)(\mathbf{A} - \mathbf{J}(t)\mathbf{C})^T + \mathbf{J}(t)\mathbf{R}\mathbf{J}(t)^T + \mathbf{Q}$$

by defining $\mathbf{J}(t) = \mathbf{J}^*(t) + \mathbf{J}(t)$ where $\mathbf{J}^*(t) = \mathbf{P}(t)\mathbf{C}^T\mathbf{R}^{-1}$.

Substituting into the equation for $\mathbf{P}(t)$ gives:

$$\begin{aligned} \frac{d\mathbf{P}(t)}{dt} &= (\mathbf{A} - \mathbf{J}(t)\mathbf{C} - \tilde{\mathbf{J}}(t)\mathbf{C})\mathbf{P}(t) + \mathbf{P}(t)(\mathbf{A} - \mathbf{J}(t)\mathbf{C} - \tilde{\mathbf{J}}(t)\mathbf{C})^T \\ &\quad + (\mathbf{J}^*(t) + \tilde{\mathbf{J}}(t))\mathbf{R}(\mathbf{J}^*(t) + \tilde{\mathbf{J}}(t))^T + \mathbf{Q} \\ &= (\mathbf{A} - \mathbf{J}(t)\mathbf{C})\mathbf{P}(t) + \mathbf{P}(t)(\mathbf{A} - \mathbf{J}(t)\mathbf{C})^T \\ &\quad + \mathbf{J}^*(t)\mathbf{R}\mathbf{J}^*(t) + \mathbf{Q} + \tilde{\mathbf{J}}(t)\mathbf{R}(\tilde{\mathbf{J}}(t))^T \end{aligned}$$

We clearly see that $\dot{\mathbf{P}}(t)$ is minimized at every time if we choose $\tilde{\mathbf{J}}(t) = 0$. Thus, $\mathbf{J}^*(t)$ is the optimal-filter gain, because it minimizes $\dot{\mathbf{P}}(t)$ (*and hence* $\mathbf{P}(t)$) for all t .

In summary, the optimal filter satisfies

$$\frac{d\hat{x}(t)}{dt} = \mathbf{A}\hat{x}(t) + \mathbf{J}^*(t)[y'(t) - \mathbf{C}\hat{x}(t)]$$

where the optimal gain $\mathbf{J}^*(t)$ satisfies

$$\mathbf{J}^*(t) = \mathbf{P}(t)\mathbf{C}^T\mathbf{R}^{-1}$$

and $\mathbf{P}(t)$ is the solution to

$$\begin{aligned} \frac{d\mathbf{P}(t)}{dt} = & (\mathbf{A} - \mathbf{J}^*(t)\mathbf{C})\mathbf{P}(t) + \mathbf{P}(t)(\mathbf{A} - \mathbf{J}^*(t)\mathbf{C})^T \\ & + \mathbf{J}^*(t)\mathbf{R}(\mathbf{J}^*(t))^T + \mathbf{Q} \end{aligned}$$

subject to $\mathbf{P}(0) = \mathbf{P}_0$.

The key design equation for $\mathbf{P}(t)$ is

$$\begin{aligned}\frac{d\mathbf{P}(t)}{dt} = & (\mathbf{A} - \mathbf{J}^*(t)\mathbf{C})\mathbf{P}(t) + \mathbf{P}(t)(\mathbf{A} - \mathbf{J}^*(t)\mathbf{C})^T \\ & + \mathbf{J}^*(t)\mathbf{R}(\mathbf{J}^*(t))^T + \mathbf{Q}\end{aligned}$$

This can also be simplified to

$$\frac{d\mathbf{P}(t)}{dt} = \mathbf{Q} - \mathbf{P}(t)\mathbf{C}^T\mathbf{R}^{-1}\mathbf{C}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^T + \mathbf{A}\mathbf{P}(t)$$

The reader will recognize that the solution to the optimal linear filtering problem presented above has a very close connection to the LQR problem presented earlier. This is not surprising in view of the duality idea mentioned earlier

Time Varying Systems ?

It is important to note, in the above derivation, that it makes no difference whether the system is time varying (*i.e.*, \mathbf{A} , \mathbf{C} , \mathbf{Q} , \mathbf{R} , *etc.* are all functions of *time*). This is often important in applications.

Properties ?

When we come to properties of the optimal filter, these are usually restricted to the time-invariant case (*or closely related cases - e.g., periodic systems*). Thus, when discussing the steady-state filter, it is usual to restrict attention to the case in which **A**, **C**, **Q**, **R**, etc. are not explicit functions of time.

The properties of the optimal filter then follow directly from the optimal LQR solutions, under the correspondences given in Table 22.10 on the next slide.

Regulator	Filter
τ	$t - \tau$
t_f	0
\mathbf{A}	$-\mathbf{A}^T$
\mathbf{B}	$-\mathbf{C}^T$
Ψ	\mathbf{Q}
Φ	\mathbf{R}
Ψ_f	\mathbf{P}_o

Table 22.1: *Duality between quadratic regulators and filters*

Note that, using the above correspondences, one can convert an optimal filtering problem into an optimal control problem and vice versa.

In particular, one is frequently interested in the steady-state optimal filter obtained when \mathbf{A} , \mathbf{C} , \mathbf{Q} and \mathbf{R} are time invariant and the filtering horizon tends to infinity. By duality with the optimal control problem, the steady-state filter takes the form

$$\frac{d\hat{x}(t)}{dt} = \mathbf{A}\hat{x} + \mathbf{J}_s^\infty (y' - \mathbf{C}\hat{x})$$

where

$$\mathbf{J}_s^\infty = \mathbf{P}_\infty^s \mathbf{C}^T \mathbf{R}^{-1}$$

and \mathbf{P}_∞^s is the stabilizing solution of the following CTARE:

$$\mathbf{Q} - \mathbf{P}_\infty \mathbf{C}^T \mathbf{R}^{-1} \mathbf{C} \mathbf{P}_\infty + \mathbf{P}_\infty \mathbf{A}^T + \mathbf{A} \mathbf{P}_\infty = 0$$

We state without proof the following facts that are the duals of those given for the LQP.

- (i) Say that the system (\mathbf{C}, \mathbf{A}) is detectable from $y(t)$; and
- (ii) Say that the system states are all perturbed by noise.
(Technically, this is stated as requiring that $(\mathbf{A}, \mathbf{Q}^{1/2})$ is stabilizable).

Then, the optimal solution of the filtering Riccati equation tends to a steady-state limit \mathbf{P}^s_∞ as $t \rightarrow \infty$. This limit has two key properties:

- ◆ \mathbf{P}^s_∞ is the only nonnegative solution of the matrix algebraic Riccati Equation

$$\mathbf{Q} - \mathbf{P}_\infty \mathbf{C}^T \mathbf{R}^{-1} \mathbf{C} \mathbf{P}_\infty + \mathbf{P}_\infty \mathbf{A}^T + \mathbf{A} \mathbf{P}_\infty = 0$$

obtained by setting $d\mathbf{P}(t)/dt$ in

$$\frac{d\mathbf{P}(t)}{dt} = \mathbf{Q} - \mathbf{P}(t) \mathbf{C}^T \mathbf{R}^{-1} \mathbf{C} \mathbf{P}(t) + \mathbf{P}(t) \mathbf{A}^T + \mathbf{A} \mathbf{P}(t)$$

-
- ◆ When this steady-state value is used to generate a steady-state observer, then the observer has the property that $(\mathbf{A} - \mathbf{J}_s \infty \mathbf{C})$ is a stability matrix.

Note that this gives conditions under which a *stable* filter can be designed. Placing the filter poles in particular regions follows the same ideas as used earlier in the case of optimal control.

Discrete-Time Optimal Quadratic Filter

We can readily develop discrete forms for the optimal filter. In particular, consider a discrete-time system having the following state space description:

$$\begin{aligned}x[k + 1] &= \mathbf{A}x[k] + \mathbf{B}u[k] + w[k] \\y[k] &= \mathbf{C}[k] + v[k]\end{aligned}$$

where $w[k] \in \mathbb{R}^n$ and $v[k] \in \mathbb{R}^n$ are uncorrelated stationary stochastic processes, with covariances given by

$$\begin{aligned}\mathcal{E}\{w[k]w^T[\zeta]\} &= \mathbf{Q}\delta_K[k - \zeta] \\ \mathcal{E}\{v[k]v^T[\zeta]\} &= \mathbf{R}\delta_K[k - \zeta]\end{aligned}$$

where $\mathbf{Q} \in \mathbb{R}^{n \times p}$ is a symmetric nonnegative definite matrix and $\mathbf{R} \in \mathbb{R}^{n \times p}$ is a symmetric positive definite matrix

Consider now the following observer to estimate the system state:

$$\hat{x}[k + 1] = \mathbf{A}\hat{x}[k] + \mathbf{B}u[k] + \mathbf{J}_o[k](y[k] - \mathbf{C}\hat{x}[k])$$

Furthermore, assume that the initial state $x[0]$ satisfies

$$\mathcal{E}\{(x[0] - \hat{x}[0])(x[0] - \hat{x}[0])^T\} = \mathbf{P}_o$$

Then the optimal choice (*in a quadratic sense*) for the observer gain sequence $\{\mathbf{J}_o[k]\}$ is given by

$$\mathbf{J}_o[k] = \mathbf{A}\mathbf{P}[k]\mathbf{C}^T (\mathbf{R} + \mathbf{C}\mathbf{P}[k]\mathbf{C}^T)^{-1}$$

where $\mathbf{P}[k]$ satisfies the following discrete-time dynamic Riccati equation (DTDRE).

$$\mathbf{P}[k + 1] = \mathbf{A}(\mathbf{P}[k] - \mathbf{P}[k]\mathbf{C}^T(\mathbf{R} + \mathbf{C}\mathbf{P}[k]\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{P}[k])\mathbf{A}^T + \mathbf{Q}$$

which can be solved forward in time, subject to

$$\mathbf{P}[0] = \mathbf{P}_o$$

The steady-state ($k \rightarrow \infty$) filter gain satisfies the DTARE given by

$$\mathbf{A}[\mathbf{P}_\infty - \mathbf{P}_\infty \mathbf{C}^T (\mathbf{R} + \mathbf{C} \mathbf{P}_\infty \mathbf{C}^T)^{-1} \mathbf{C} \mathbf{P}_\infty] \mathbf{A}^T + \mathbf{Q} = \mathbf{P}_\infty$$

Stochastic Noise Models

In the above development, we have simply represented the noise as a white-noise sequence ($\{\omega(k)\}$) and a white measurement-noise sequence ($\{\nu(k)\}$). Actually, this is much more general than it may seem at first sight. For example, it can include colored noise having an arbitrary rational noise spectrum. The essential idea is to model this noise as the output of a linear system (*i.e.*, a *filter*) driven by white noise.

Thus, say that a system is described by

$$\begin{aligned}x(k+1) &= \mathbf{A}x(k) + \mathbf{B}u(k) + \omega_c(k) \\y(k) &= \mathbf{C}x(k) + v(k)\end{aligned}$$

where $\{\omega_c(k)\}$ represents *colored noise* - noise that is white noise passed through a filter. Then we can add the additional noise model to the description. For example, let the noise filter be

$$\begin{aligned}x'(k+1) &= \mathbf{A}'x(k) + \omega(k) \\ \omega_c(k) &= \mathbf{C}'x'(k)\end{aligned}$$

where $\{\omega(k)\}$ is a white-noise sequence.

This yields a composite system *driven by white noise*, of the form

$$\begin{aligned}\bar{x}(k+1) &= \bar{\mathbf{A}}\bar{x}(k) + \bar{\mathbf{B}}u(k) + \bar{\omega}(k) \\ y(k) &= \bar{\mathbf{C}}\bar{x}(k) + v(k)\end{aligned}$$

where

$$\begin{aligned}\bar{x}(k) &= [x(k)^T, x'(k)^T]^T \\ \bar{\omega}(k) &= [0, \omega(k)^T]^T\end{aligned}$$

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{C}' \\ 0 & \mathbf{A}' \end{bmatrix}; \quad \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}; \quad \bar{\mathbf{C}} = [\mathbf{C} \quad 0]$$

Because of the importance of the discrete Kalman Filter in applications, we will repeat below the formulation and derivation. The discrete derivation may be easier to follow than the continuous case given earlier.

Discrete-Time State-Space Model

$$x_{k+1} = \mathbf{A}x_k + Bu_k$$

$$y_k = \mathbf{C}x_k$$

The above state-space system is deterministic since no noise is present.

We can introduce uncertainty into the model by adding *noise* terms

$$x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k + w_k$$

Process noise ←

$$y_k = \mathbf{C}x_k + n_k$$

Measurement noise ←

This is referred to as a stochastic state-space model.

In particular, for a 3rd Order System we have:

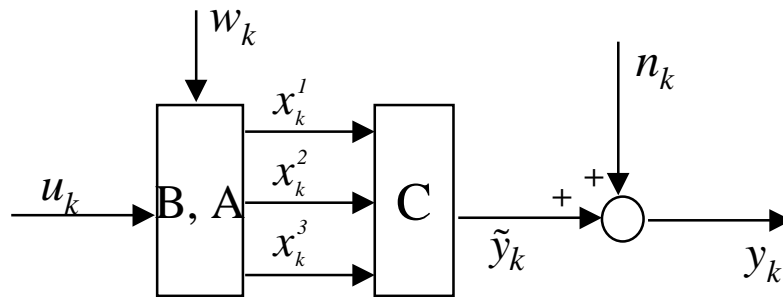
$$x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k + w_k \quad \leftarrow \begin{array}{l} \text{Process} \\ \text{noise} \end{array}$$

$$y_k = \mathbf{C}x_k + n_k \quad \leftarrow \begin{array}{l} \text{Measurement} \\ \text{noise} \end{array}$$

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}_{k+1} = \begin{pmatrix} \mathbf{A}^{11} & \mathbf{A}^{12} & \mathbf{A}^{13} \\ \mathbf{A}^{21} & \mathbf{A}^{22} & \mathbf{A}^{23} \\ \mathbf{A}^{31} & \mathbf{A}^{32} & \mathbf{A}^{33} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}_k + \begin{pmatrix} \mathbf{B}^1 \\ \mathbf{B}^2 \\ \mathbf{B}^3 \end{pmatrix} u_k + \begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix}_k$$

$$(y_k) = (\mathbf{C}^1 \quad \mathbf{C}^2 \quad \mathbf{C}^3) \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}_k + (n_k)$$

This is illustrated below:



X_k - state vector

A - system matrix

B - input matrix

C - output matrix

y_k - output (PV_m)

\tilde{y}_k - noise free output (PV)

w_k - process noise

n_k - measurement noise

u_k - control input (MV)

We recall that a Kalman Filter is a particular type of observer. We propose a form for this observer on the next slide.

Observers

We are interested in constructing an optimal observer for the following state-space model:

$$x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k + w_k$$

$$y_k = \mathbf{C}x_k + n_k$$

An observer is constructed as follows:

$$\hat{x}_{k+1} = \mathbf{A}\hat{x}_k + \mathbf{B}u_k + J (y_k - \hat{y}_k)$$

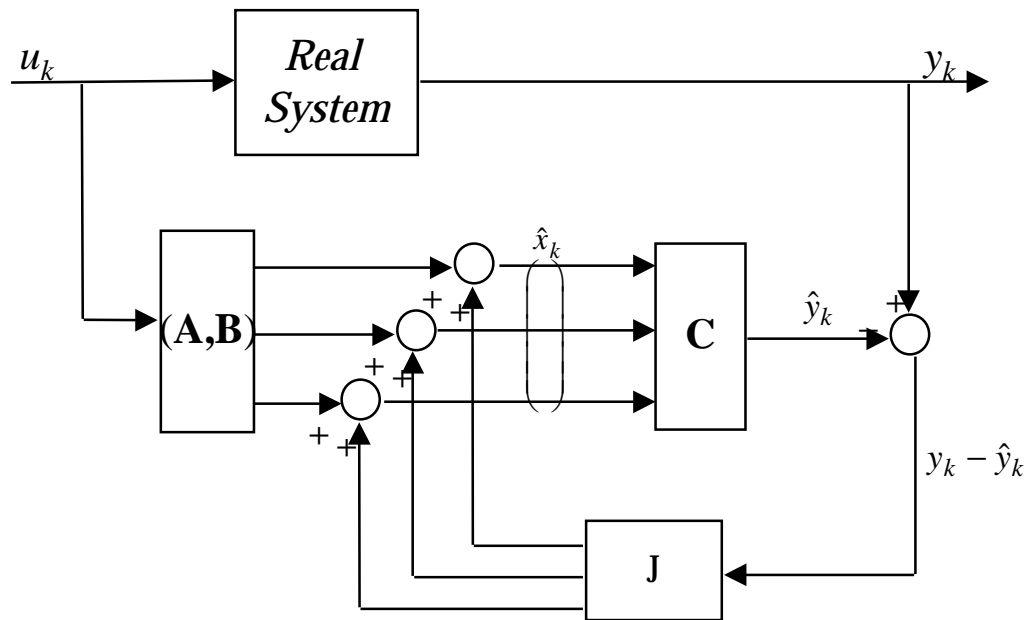
where J is the observer gain vector, and \hat{y}_k is the best estimate of y_k i.e. $\hat{y}_k = \mathbf{C}\hat{x}_k$.

Thus the observer takes the form:

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}\hat{\mathbf{x}}_k + \mathbf{B}u_k + J (y_k - \mathbf{C}\hat{\mathbf{x}}_k)$$

This equation can also be written as:

$$\hat{\mathbf{x}}_{k+1} = (\mathbf{A} - J\mathbf{C})\hat{\mathbf{x}}_k + Jy_k + \mathbf{B}u_k$$



Observer in Block Diagram Form

Kalman Filter

The Kalman filter is a special observer that has *optimal* properties under certain hypotheses. In particular, suppose that.

- 1) w_k and n_k are statistically independent
(*uncorrelated in time and with each other*)
- 2) w_k and n_k , have Gaussian distributions
- 3) The system is known exactly

The Kalman filter algorithm provides an observer vector J that results in an optimal state estimate.

The optimal J is referred to as the Kalman Gain (J^*)

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}\hat{\mathbf{x}}_k + \mathbf{B}u_k + J^*(y_k - \hat{y}_k)$$

$$\hat{y} = \mathbf{C}\hat{\mathbf{x}}_k$$

Five step Kalman Filter Derivation

Background:

$E[\bullet]$ - Expected Value or Average

$$\Sigma_w^2 = \text{cov}(w_k) = E[w_k w_k^T] \quad w_k \text{ - vector}$$

$$(\text{scalar} : \sigma_w^2 = \text{var}(w_k) = E[w_k^2]) \quad \Sigma_w^2 \text{ - matrix}$$

$$\Sigma_n^2 = \text{cov}(n_k) = E[n_k n_k^T] \quad n_k \text{ - vector}$$

$$(\text{scalar} : \sigma_n^2 = \text{var}(n_k) = E[n_k^2]) \quad \Sigma_n^2 \text{ - matrix}$$

The above assumes w_k and n_k are zero mean. Σ_w^2 and Σ_n^2 are usually diagonal. Σ_w^2 and Σ_n^2 are matrix versions of standard deviation squared or variance.

Step 1:

Given

$$x_{k+1} = \mathbf{A}x_k + w_k$$

$$E[x_0 x_0^T] = P_0$$

$$E[w_k w_k^T] = \Sigma_w^2$$

Calculate

$$P_k = E[x_k x_k^T]$$

Solution:

$$\begin{aligned} E \left[x_{k+1} x_{k+1}^T \right] &= E \left[(\mathbf{A}x_k + w_k)(\mathbf{A}x_k + w_k)^T \right] \\ &= E \left[(\mathbf{A}x_k + w_k)(x_k^T \mathbf{A}^T + w_k^T) \right] \\ &= E \left[(\mathbf{A}x_k x_k^T \mathbf{A}^T) + (\mathbf{A}x_k w_k^T) + (w_k x_k^T \mathbf{A}^T) + (w_k w_k^T) \right] \\ &= \mathbf{A}E \left[x_k x_k^T \right] \mathbf{A}^T + E \left[\mathbf{A}x_k w_k^T \right] + E \left(w_k x_k^T \mathbf{A}^T \right) + E \left[w_k w_k^T \right] \\ &= \mathbf{A} \mathbf{P}_k \mathbf{A}^T + 0 + 0 + \sum_w^2 \end{aligned}$$

$$\mathbf{P}_{k+1} = \mathbf{A} \mathbf{P}_k \mathbf{A}^T + \sum_w^2$$

Step 2:

$$x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k + w_k$$

$$y_k = \mathbf{C}x_k + n_k$$

What is a good estimate of x_k ?

We try the following form for the filter (*where the sequence $\{J_k\}$ is yet to be determined*):

$$\hat{x}_{k+1} = \hat{\mathbf{A}}\hat{x}_k + \mathbf{B}u_k + J_k (y_k - \mathbf{C}\hat{x}_k)$$

Step 3:

Given

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}u_k + w_k$$

$$y_k = \mathbf{C}\mathbf{x}_k + n_k$$

and

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}\hat{\mathbf{x}}_k + \mathbf{B}u_k + J_k (y_k - \mathbf{C}\hat{\mathbf{x}}_k)$$

Evaluate:

$$\text{cov}(\mathbf{x}_k - \hat{\mathbf{x}}_k) = E \left[(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T \right]$$

Solution:

$$\begin{aligned}\tilde{\mathbf{x}}_{k+1} &= \mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1} \\ &= \mathbf{A}\mathbf{x}_k + \mathbf{B}u_k + w_k - (\mathbf{A}\mathbf{x}_k + \mathbf{B}u_k + J_k y_k - J_k \mathbf{C}\hat{\mathbf{x}}_k) \\ &= \mathbf{A}\tilde{\mathbf{x}}_k + w_k - J_k (\mathbf{C}\mathbf{x}_k + n_k) + J_k \mathbf{C}\hat{\mathbf{x}}_k \\ &= \mathbf{A}\tilde{\mathbf{x}}_k - J_k \mathbf{C}\tilde{\mathbf{x}}_k + w_k - J_k n_k \\ &= (\mathbf{A} - J_k \mathbf{C})\tilde{\mathbf{x}}_k + w_k - J_k n_k\end{aligned}$$

Let

$$\mathbf{P}_{k+1} = E \left[\tilde{\mathbf{x}}_{k+1} \tilde{\mathbf{x}}_{k+1}^T \right]$$

Then applying the result of step 2 we have

$$\mathbf{P}_{k+1} = (\mathbf{A} - \mathbf{J}_k \mathbf{C}) \mathbf{P}_k (\mathbf{A} - \mathbf{J}_k \mathbf{C})^T + \sum_w^2 + \mathbf{J}_k \sum_n^2 \mathbf{J}_k^T$$

Step 4:

Given

$$\mathbf{P}_k = E \left[\tilde{\mathbf{x}}_k \tilde{\mathbf{x}}_k^T \right]$$

Evolves according to

$$\mathbf{P}_{k+1} = (\mathbf{A} - J_k \mathbf{C}) \mathbf{P}_k (\mathbf{A} - J_k \mathbf{C})^T + \sum_w^2 + J_k \sum_n^2 J_k^T$$

What is the best (*optimal*) value for J (*call it J_k^**)?

Solution:

Since \mathbf{P}_{k+1} is quadratic in J_k , it seems we should be able to determine J_k so as to minimize \mathbf{P}_{k+1} .

We first consider the scalar case.

The equation for \mathbf{P}_{k+1} then takes the form

$$p_{k+1} = (a - j_k c)^2 p_k + \sigma_w^2 + j_k^2 \sigma_n^2$$

Differentiate with respect to j_k

$$\frac{\partial p_{k+1}}{\partial j_k} = -2(a - j_k c) c p_k + 2 j_k \sigma_n^2$$

$$0 = -(a - j_k c) p_k c + j_k \sigma_n^2$$

Hence

$$j_k^* = a p_k C (C p_k C + \sigma_n^2)^{-1}$$

Also p_k evolves according to the equation on the top of the slide with j_k replaced by the optimal value j_k^* .

The corresponding Matrix version is

$$J_k = J_k^* = \mathbf{A} \mathbf{P}_k \mathbf{C}^T (\mathbf{C} \mathbf{P}_k \mathbf{C}^T + \Sigma_n^2)^{-1}$$

Step 5:

Bring it all together.

Given

$$x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k + w_k$$

$$y_k = \mathbf{C}x_k + n_k$$

where

$$\Sigma_w^2 = E[w_k w_k^T]$$

$$\Sigma_n^2 = E[n_k n_k^T]$$

$$P_0 = E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T]$$

$$\hat{x}_0 = \text{Initial state estimate}$$

Find optimal filter.

Solution:

The Kalman Filter

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{A}}\hat{\mathbf{x}}_k + \mathbf{B}u_k + \mathbf{J}_k^* (y_k - \mathbf{C}\hat{\mathbf{x}}_k)$$

$$\mathbf{J}_k^* = \mathbf{A}\mathbf{P}_k\mathbf{C}^T (\mathbf{C}\mathbf{P}_k\mathbf{C}^T + \Sigma_n^2)^{-1}$$

$$\begin{aligned} \mathbf{P}_{k+1} &= (\mathbf{A} - \mathbf{J}_k^* \mathbf{C}) \mathbf{P}_k (\mathbf{A} - \mathbf{J}_k^* \mathbf{C})^T + \Sigma_w^2 + \mathbf{J}_k^* \Sigma_n^2 \mathbf{J}_k^{*T} \\ &= \mathbf{A} \left(\mathbf{P}_k - \mathbf{P}_k \mathbf{C}^T (\mathbf{C} \mathbf{P}_k \mathbf{C}^T + \Sigma_n^2)^{-1} \mathbf{C} \mathbf{P}_k \right) \mathbf{A}^T + \Sigma_w^2 \end{aligned}$$

Simple Example

Problem:

Estimate a constant from measurements y_k corrupted by white noise of variance 1.

Model for constant $\Rightarrow x_{k+1} = x_k; w_k = 0$

Model for the corrupted measurement $\Rightarrow y_k = x_k + n_k$

$$\Rightarrow E[n_k^2] = \text{var}(n_k) = \sum_n^2 = 1$$

An initial estimate of this constant is given, but this initial estimate has a variance of 1 around the true value.

Solution Formulation

$$E[(x_0 - \hat{x}_0)^2] = \text{var}(x_0 - \hat{x}_0) = \mathbf{P}_0 = 1$$

$$\mathbf{A} = 1; \quad \mathbf{B} = 0; \quad \Sigma_w^2 = 0; \quad \Sigma_n^2 = 1$$

From previous Kalman Filter equations with $\mathbf{A} = 1; \mathbf{B} = 0;$
 $\mathbf{C} = 1; \Sigma_w^2 = 0; \Sigma_n^2 = 1$

$$\hat{x}_{k+1} = \hat{x}_k + J_k^* (y_k - \hat{x}_k)$$

$$J_k^* = \frac{\mathbf{P}_k}{\mathbf{P}_k + 1}$$

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \frac{\mathbf{P}_k^2}{\mathbf{P}_k + 1}$$

Calculate P_k (Given $P_0 = 1$)

$$P_1 = P_0 - \frac{P_0^2}{P_0 + 1} = 1 - \frac{1^2}{1+1} = \frac{1}{2}$$

$$P_2 = P_1 - \frac{P_1^2}{P_1 + 1} = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{\frac{1}{2} + 1} = \frac{1}{3}$$

$$P_3 = \frac{1}{4}, \quad P_4 = \frac{1}{5}, \quad P_5 = \frac{1}{6}; \quad \text{etc.}$$

Calculate the estimate \hat{x}_k given the initial estimate \hat{x}_0 and the noisy measurements y_k

$$\begin{aligned}\hat{x}_1 &= \hat{x}_0 + \frac{\mathbf{P}_0}{\mathbf{P}_0 + 1} (y_0 - \hat{x}_0) \\ &= \hat{x}_0 + \frac{1}{1+1} (y_0 - \hat{x}_0) \\ &= \frac{1}{2} (\hat{x}_0 + y_0)\end{aligned}$$

$$\begin{aligned}\hat{x}_2 &= \hat{x}_1 + \frac{\mathbf{P}_1}{\mathbf{P}_1 + 1} (y_1 - \hat{x}_1) \\ &= \frac{1}{2} (\hat{x}_0 + y_0) + \frac{\frac{1}{2}}{\frac{1}{2} + 1} \left(y_1 - \frac{1}{2} (\hat{x}_0 + y_0) \right) \\ &= \frac{1}{3} (\hat{x}_0 + y_0 + y_1) \\ \hat{x}_3 &= \frac{1}{4} (\hat{x}_0 + y_0 + y_1 + y_2) \\ \hat{x}_4 &= \frac{1}{5} (\hat{x}_0 + y_0 + y_1 + y_2 + y_3)\end{aligned}$$

The above result (*for this special problem*) is intuitively reasonable. Note that the Kalman Filter has simply averaged the measurements and has treated the initial estimate as an extra piece of information (*like an extra measurement*). This is probably the answer you would have guessed for estimating the constant before you ever heard of the Kalman Filter.

The fact that the answer is heuristically reasonable in this special case encourages us to believe that the Kalman Filter may give a good solution in other, more complex cases. Indeed it does !

State-Estimate Feedback

Finally, we can combine the state estimation provided by the Kalman Filter with the state-variable feedback determined earlier to yield the following state-estimate feedback-control law:

$$u(t) = -\mathbf{K}\hat{x}(t) + \bar{r}(t)$$

Note that the closed-loop poles resulting from the use of this law are the union of the eigenvalues that result from the use of the state feedback together with the eigenvalues associated with the observer. Actually, the result can also be shown to be optimal via Stochastic Dynamic Programming. (*However, this is beyond the scope of the treatment presented here*).

Achieving Integral Action in LQR Synthesis

An important aspect not addressed so far is that optimal control and optimal state-estimate feedback do not automatically introduce integral action. The latter property is an architectural issue that has to be *forced* onto the solution.

One way of forcing integral action is to put a set of integrators at the output of the plant.

This can be described in state space form as

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$

$$y(t) = \mathbf{C}x(t)$$

$$\dot{z}(t) = -y(t)$$

As before, we can use an observer (*or Kalman filter*) to estimate x from u and y . Hence, in the sequel we will assume (*without further comment*) that x and z are directly measured. The composite system can be written in state space form as

$$\dot{x}'(t) = \mathbf{A}'x(t) + \mathbf{B}'u(t)$$

Where

$$x' = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}; \quad \mathbf{A}' = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix}; \quad \mathbf{B}' = \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}$$

We then determine state feedback (from $x'(t)$) to stabilize the composite system.

The final architecture of the control system would then appear as below.

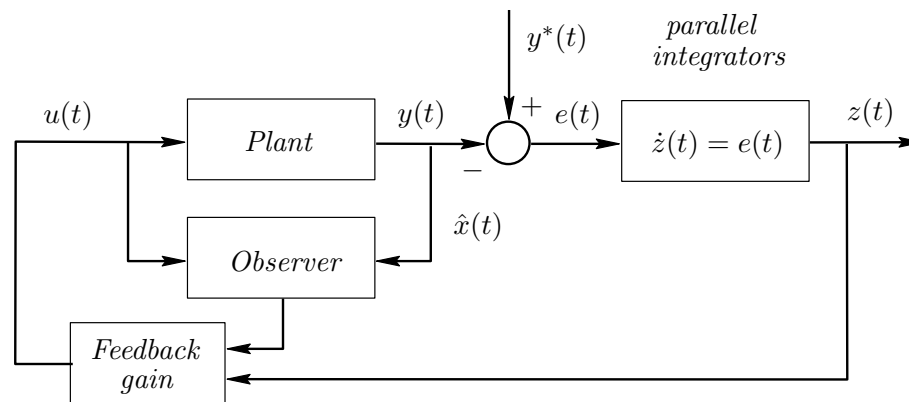


Figure 22.2: *Integral action in MIMO control*

Industrial Applications

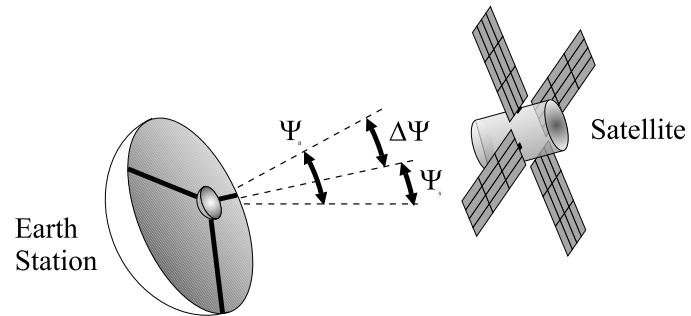
Multivariable design based on LQR theory and the Kalman filter accounts for thousands of real-world applications.

The key issue in using these techniques in practice lies in the problem formulation; once the problem has been properly posed, the solution is usually rather straightforward. Much of the success in applications of this theory depends on the formulation, so we will conclude this chapter with brief descriptions of four real-world applications.

Geostationary Satellite Tracking

It is known that so-called geostationary satellites actually appear to *wobble* in the sky. The period of this wobble is one sidereal day. If one wishes to point a receiving antenna exactly at a satellite so as to maximize the received signal, then it is necessary to track this perceived motion. The required pointing accuracy is typically to within a few hundredths of a degree. The physical set-up is as shown in the next figure.

Figure 22.4: *Satellite and antenna angle definitions*



One could use an open-loop solution to this problem, as follows: Given a model (*e.g.*, *a list of pointing angles versus time*), the antenna could be pointed in the correct orientation as indicated by position encoders. This technique is used in practice, but it suffers from the following practical issues:

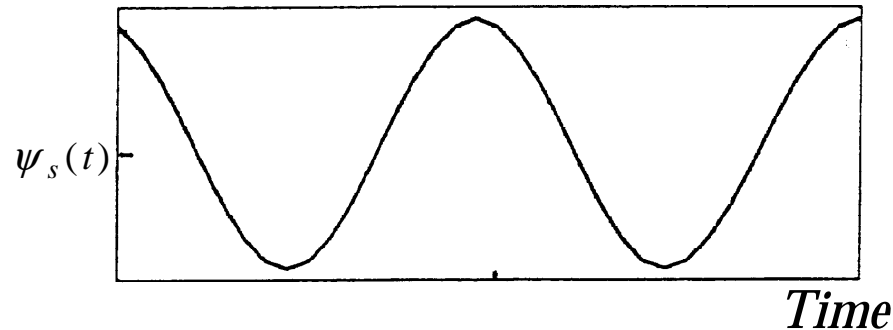
- ◆ It requires high absolute accuracy in the position encoders, antenna, and reflector structure.
- ◆ It also requires regular *maintenance* to put in new model parameters
- ◆ It cannot compensate for wind, thermal, and other time-varying effects on the antenna and reflector.

This motivates the use of a closed-loop solution. In such a solution, the idea is to move the antenna periodically so as to find the direction of maximum signal strength. However, the data so received are noisy for several reasons, including the following:

- ◆ noise in the received signal, p ;
- ◆ variations in the signal intensity transmitted from the satellite;
- ◆ imprecise knowledge of the beam pattern for the antenna; and
- ◆ the effect of wind gusts on the structure and the reflector.

It is a reasonable hypothesis that we can smooth this data by using a Kalman filter. Toward this end, we need first to build a model for the orbit. Now, as seen from the earth, the satellite executes a periodic motion in the two axes of the antenna (*azimuth and elevation - see next slide*). Several harmonics are present but the dominant harmonic is the fundamental. This leads to a model of the form $y(t) = \Psi_s(t) = x_1 + x_2 \sin \omega t + x_3 \cos \omega t$ where $\Psi_s(t)$ is, say, the azimuth angle as a function of time. The frequency ω in this application is known. There are several ways of describing this model in state space form.

Typical satellite motion is close to periodic, with a period of 1 sidereal day:



Typical inclined orbit satellite motion

Linear Model:

Several Harmonics are present, but the dominant harmonic is the fundamental:

$$\begin{aligned}\psi_s(t) &\approx x_1 + x_2 \sin(\omega t) + x_3 \cos(\omega t) \\ &= C(t)x\end{aligned}$$

with

$$C(t) = \begin{bmatrix} 1 & \sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

$$x^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

This can be expressed in state space form as follows:

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$y(t) = C(t)x(t)$$

where $C(t) = [1, \sin \omega t, \cos \omega t]$

Problem Reformulation:

Given noisy measurements, $y(t)$, fit a model for the unknown parameters x_1 , x_2 and x_3 .

This system is time-varying (*actually periodic*). We can then immediately apply the Kalman filter to estimate x_1 , x_2 and x_3 from noisy measurements of $y(t)$.

In practice, it is important to hypothesise the existence of a small amount of fictitious process noise which is added to the model equations. This represents the practical fact that the model is imprecise. This leads to a filter which is *robust* to the model imprecision.

One can formally derive properties of the resulting filter. Heuristically one would expect:

- ◆ As one increases the amount of hypothesised model error, the filter pays more attention to the measurements, i.e. the filter gain increases;
- ◆ As one decreases the amount of hypothesised model error, the filter pays more attention to the model. In particular, the filter will ultimately ignore the measurements after an initial transient if one assumes no model error.

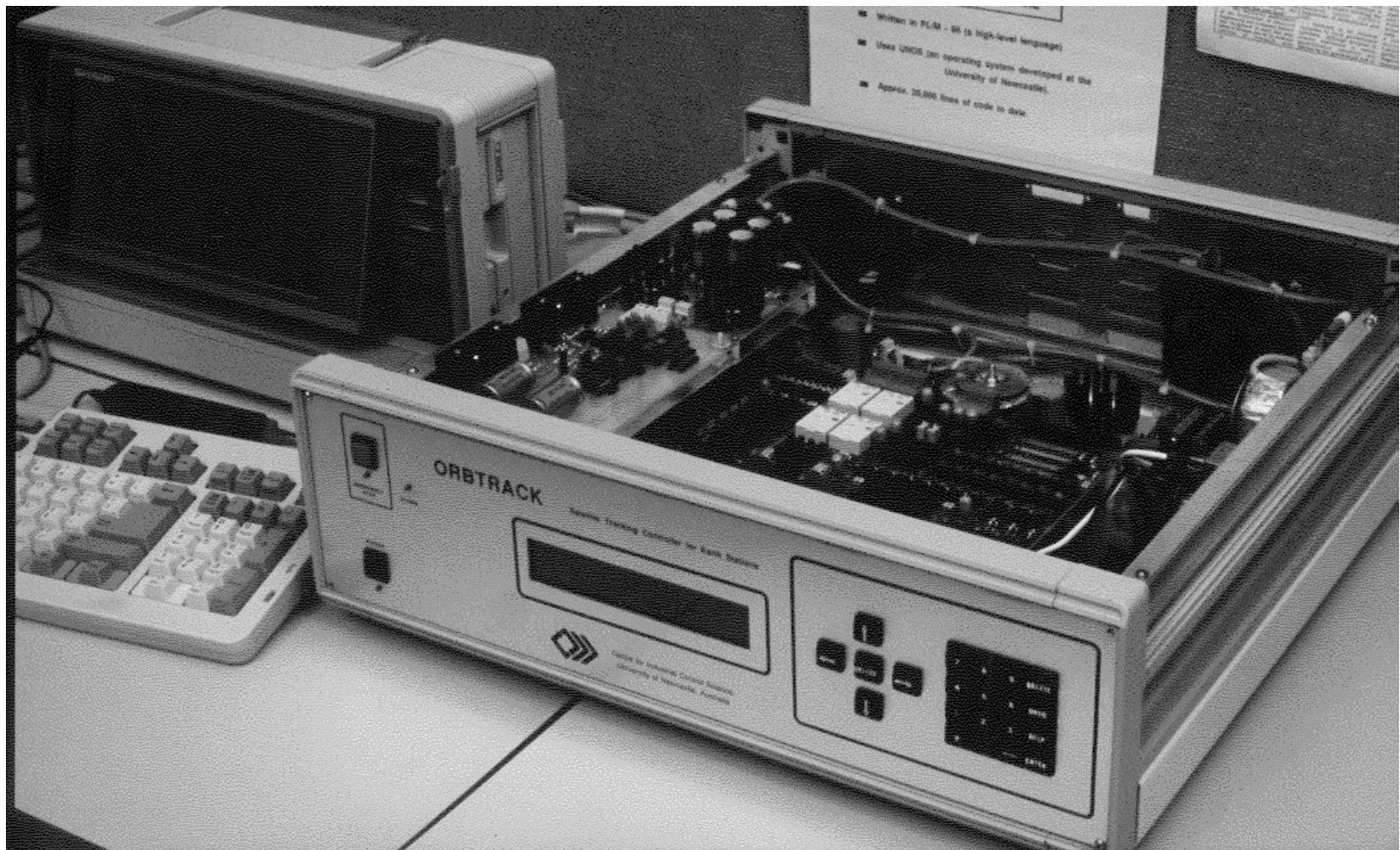
The above heuristic ideas can, in fact, be formally established.

To initialize the filter one needs;

- ◆ a guess at the current satellite orientation;
- ◆ a guess at the covariance of the initial state error ($P(0)$);
- ◆ a guess at the measurement-noise intensity (R); and
- ◆ a rough value for the added process noise intensity (Q).

A commercial system built around the above principles has been designed and built at the University of Newcastle, Australia. This system is marketed under the trade name ORBTRACK[®] and has been used in many real-world applications ranging from Australia to Indonesia and Antarctica. See next slide for photo.

ORBTRACK

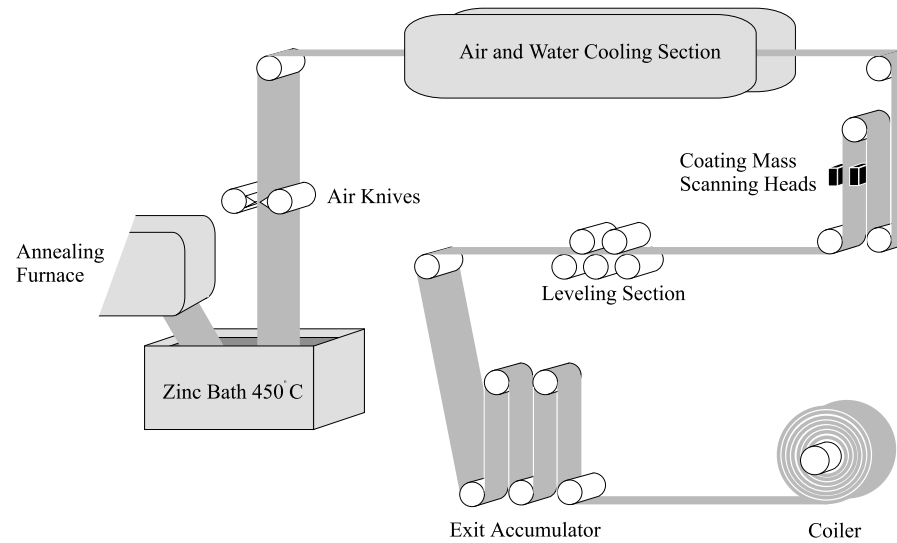


Zinc Coating-Mass Estimation in Continuous Galvanizing Lines

A diagram of a continuous galvanizing line is shown on the next slide. An interesting feature of this application is that the sheet being galvanized is a meter or so wide and many hundreds of meters long.

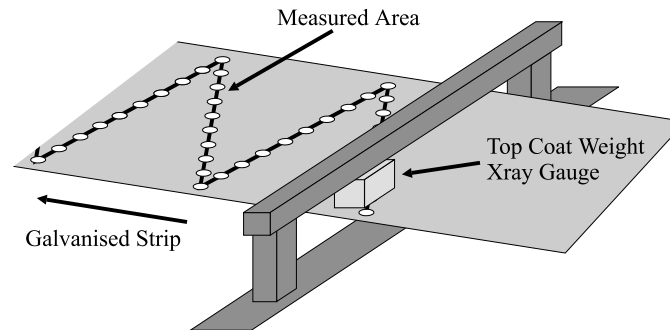
The strip passes through a zinc pot (*as in the figure*). Subsequently, excess zinc is removed by air knives. The strip then moves through a cooling section, and finally the coating mass is measured by a traversing X-ray gauge.

Figure 22.5: *Schematic diagram of continuous galvanizing line*



The x ray gauge moves backwards and forwards across the moving strip as shown diagrammatically on the next slide.

Figure 22.6: *Traversing X-ray gauge*



If one combines the lateral motion of the X-ray gauge with the longitudinal motion of the strip, then one obtains the zig-zag measurement pattern shown below.

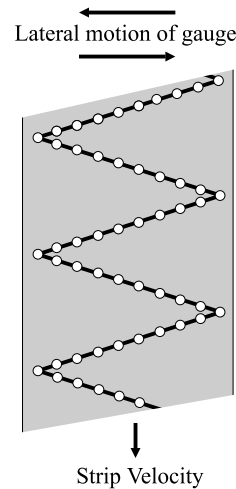


Figure 22.7:
Zig-zag measurement pattern

Because of the sparse measurement pattern, it is highly desirable to smooth and interpolate the coating-mass measurements. The Kalman filter is a possible tool to carry out this data-smoothing function. However, before we can apply this tool, we need a model for the relevant components in the coating-mass distribution. The relevant components include the following:

-
- ◆ **Shape Disturbances** (*arising from shape errors in the rolling process*).

These can be described by band-pass-filtered noise components, by using a model of the form

$$\dot{x}_1 = -\omega_1 x_1 - \left[\frac{\omega_2 \omega_1}{\omega_2 - \omega_1} \right] n$$

$$\dot{x}_2 = -\omega_2 x_2 - \left[\frac{\omega_2^2}{\omega_1 - \omega_2} \right] n$$

$$y_{sd} = (1, 1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

-
- ◆ **Cross Bow** (*a quadratic term arising from nonuniform coating effects*).

This is a quadratic function of distance across the strip and is modeled by

$$\begin{aligned}\dot{x}_3 &= 0 \\ y_{cb} &= \{d(t)[d(t) - W]\}x_3\end{aligned}$$

where $d(t)$ denotes the distance from the left edge of the strip and W denotes the total strip width.

- ◆ **Skew** (*due to misalignment of the knife jet*)

This is a term that increases linearly with distance from the edge. It can thus be modeled by

$$\dot{x}_4 = 0$$

$$y_{sc} = \{d(t)\}x_4$$

- ◆ **Eccentricity** (*due to out-of-round in the rolls*)

Say that the strip velocity is v_s and that the roll radius is r . Then this component can be modeled as

$$\dot{x}_5 = 0$$

$$\dot{x}_6 = 0$$

$$y_e = \left\{ \sin \left(\frac{v_s}{r} \right) t, \cos \left(\frac{v_s}{r} \right) t \right\} \begin{pmatrix} x_5 \\ x_6 \end{pmatrix}$$

-
- ◆ **Strip Flap** (*due to lateral movement of the strip in the vertical section of the galvanizing line*)

Let $f(t)$ denote the model for the flap; then this component is modeled by

$$\dot{x}_7 = 0$$

$$y_f = \{f(t)\}x_7$$

-
- ◆ **Mean Coating Mass** (*the mean value of the zinc layer*)

This can be simply modeled by

$$\dot{x}_8 = 0$$

$$y_m = x_8$$

Putting all of the equations together gives us an 8th-order model of the form

$$\dot{x} = \mathbf{A}x + \mathbf{B}n$$

$$z = y = \mathbf{C}(t)x + v$$

$$\mathbf{A} = \begin{bmatrix} -\omega_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\omega_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} -\left(\frac{\omega_2\omega_1}{\omega_2-\omega_1}\right) \\ -\left(\frac{\omega_2^2}{\omega_1-\omega_2}\right) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = [1, 1, d(t)[d(t) - W], d(t), \sin\left(\frac{v_s}{d}t\right), \cos\left(\frac{v_s}{d}t\right), f(t), 1]$$

Given the above model, one can apply the Kalman filter to estimate the coating-thickness model. The resultant model can then be used to interpolate the thickness measurement. Note that here the Kalman filter is actually periodic, reflecting the periodic nature of the X-ray traversing system.

A practical form of this algorithm is part of a commercial system for Coating-Mass Control developed in collaboration with the authors of this book by a company (*Industrial Automation Services Pty. Ltd.*). The following slides are taken from commercial literature describing this Coating Mass Control system.

INDUSTRIAL AUTOMATION SERVICES



COMIAC
COATING MASS CONTROL

CO AC



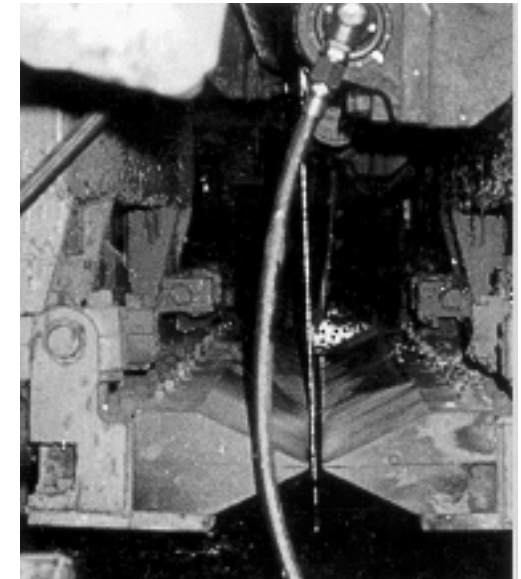
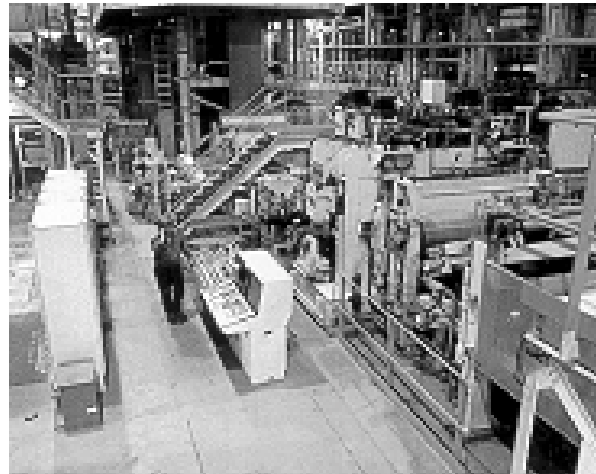
Industrial Automation Services Pty Ltd

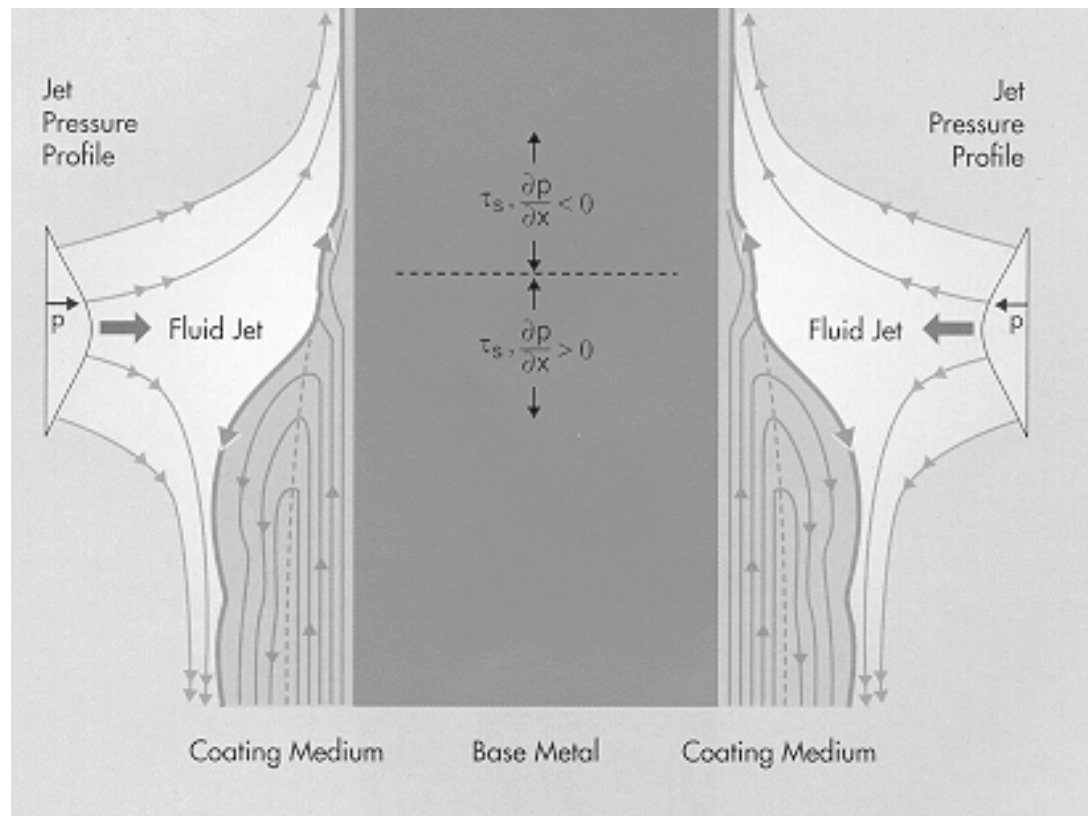
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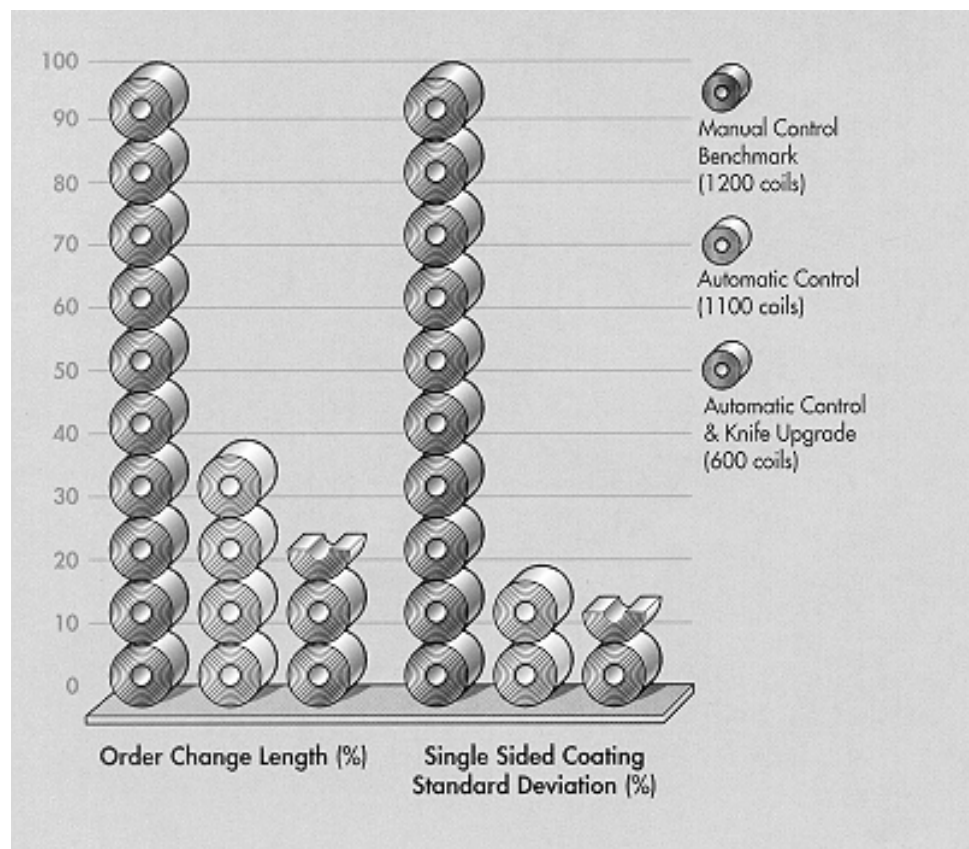
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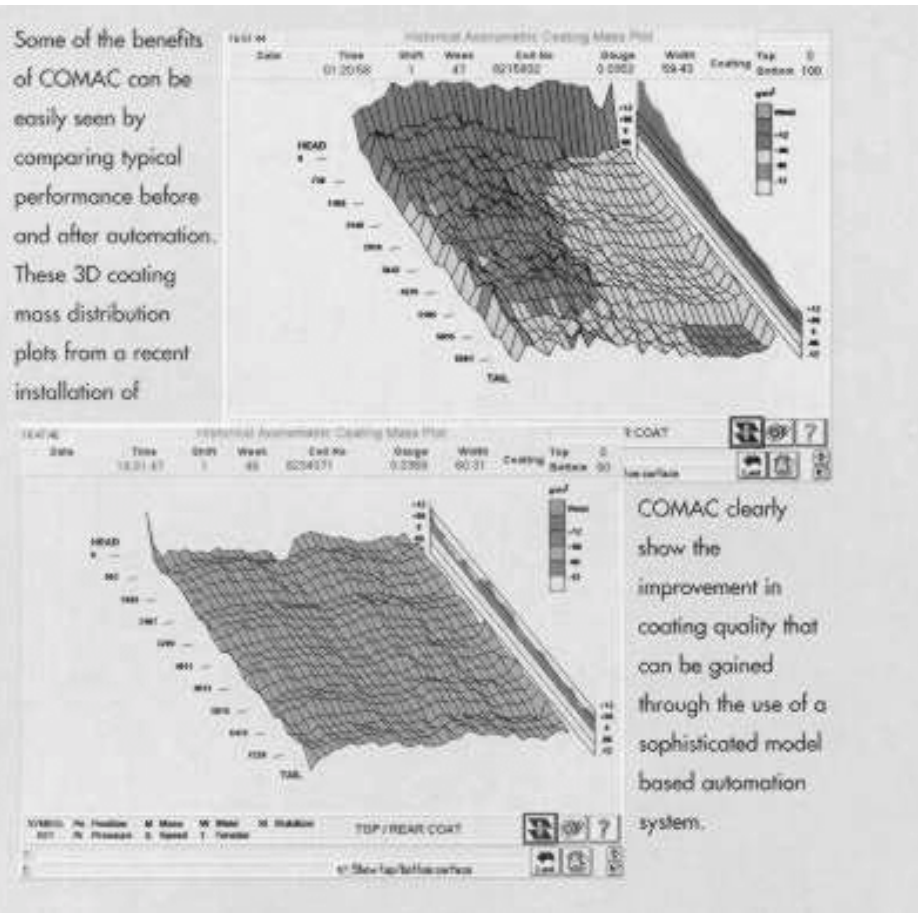
Email: info@indauto.com.au www.indauto.com.au

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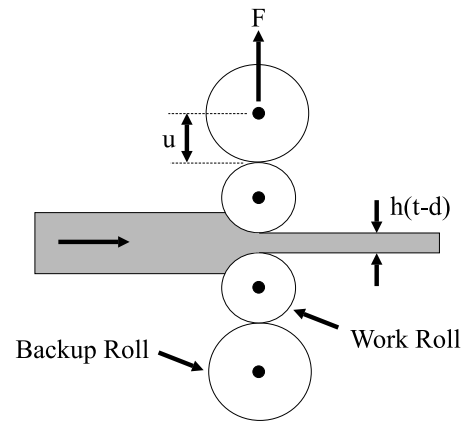




Roll-Eccentricity Compensation in Rolling Mills

The reader will recall that rolling-mill thickness-control problems were described in Chapter 8. A schematic of the set-up is shown below.

Figure 22.8: Rolling-mill thickness control



$F(t)$: Force

$h(t)$: Exit-thickness Measurement

$u(t)$: Unloaded Roll Gap (*the control variable*)

In Chapter 8, it was argued that the following virtual sensor (*called a BISRA gauge*) could be used to estimate the exit thickness and thus eliminate the transport delay from mill to measurement.

$$\hat{h}(t) = \frac{F(t)}{M} + u(t)$$

However, one difficulty that we have not previously mentioned with this virtual sensor is that the presence of eccentricity in the rolls significantly affects the results.

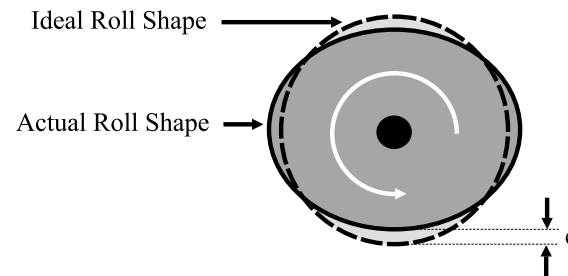


Figure 22.9: *Roll eccentricity*

To illustrate why this is so, let e denote the roll eccentricity. Then the true roll force is given by

$$F(t) = M(h(t) - u(t) + e(t))$$

In this case, the previous estimate of the thickness obtained from the force actually gives

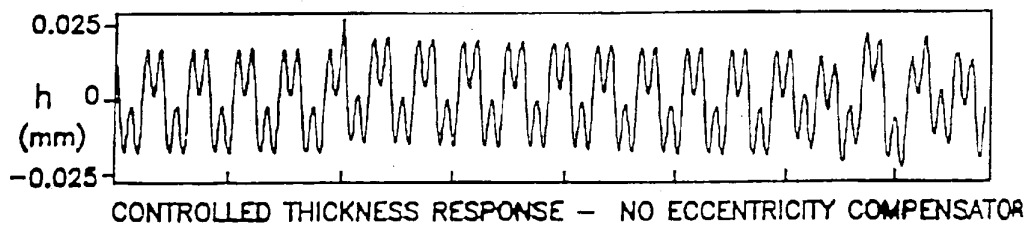
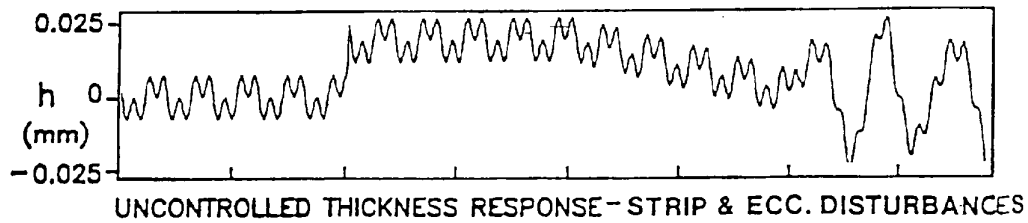
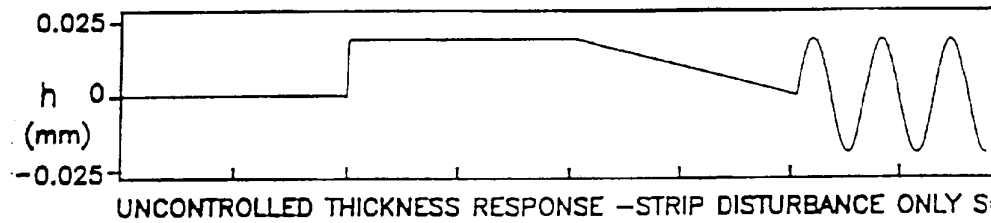
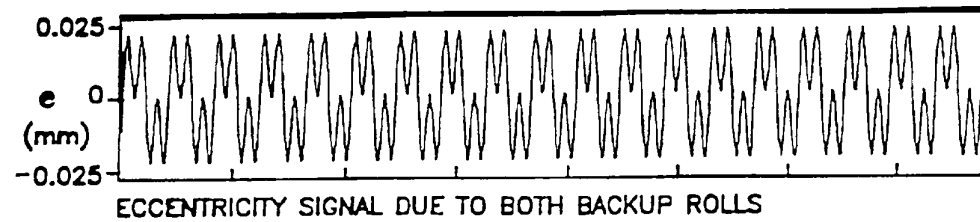
$$\hat{h}(t) = h(t) + e(t)$$

Thus, $e(t)$ represents an error, or disturbance term, in the virtual sensor output, one due to the effects of eccentricity.

This eccentricity component significantly degrades the performance of thickness control using the BISRA gauge. Thus, there is strong motivation to attempt to remove the eccentricity effect from the estimated thickness provided by the BISRA gauge.

The next slide shows a simulation which demonstrates the effect of eccentricity on the performance of a thickness control system in a rolling mill when eccentricity components are present.

- ◆ The upper trace shows the eccentricity signal
- ◆ The second top trace shows another disturbance
- ◆ The third top trace shows the effect of eccentricity in the absence of feedback control
- ◆ The bottom trace shows that when the eccentricity corrupted BISRA gauge estimate is used in a feedback control system, then the eccentricity effect is magnified.



A key property that allows us to make progress on the problem is that $e(t)$ is actually (*almost*) periodic, because it arises from eccentricity in the four rolls of the mill (*two work rolls and two back-up rolls*).

Also, the roll angular velocities are easily measured in this application by using position encoders. From this data, one can determine a multi-harmonic model for the eccentricity, of the form

$$e(t) = \sum_{k=1}^N \alpha_k \sin \omega_k t + \beta_k \cos \omega_k t$$

Each sinusoidal input can be modeled by a second order state space model of the form

$$\begin{aligned}\dot{x}_1^k(t) &= \omega_k x_2^k(t) \\ \dot{x}_2^k(t) &= -\omega_k x_1^k(t)\end{aligned}$$

Finally, consider any given measurement, say the force $F(t)$. We can think of $F(t)$ as comparing the above eccentricity components buried in noise:

$$y(t) = F(t) = \sum_{k=1}^N x_1^k(t) + x_2^k(t) + n(t)$$

We can then apply the Kalman filter to estimate

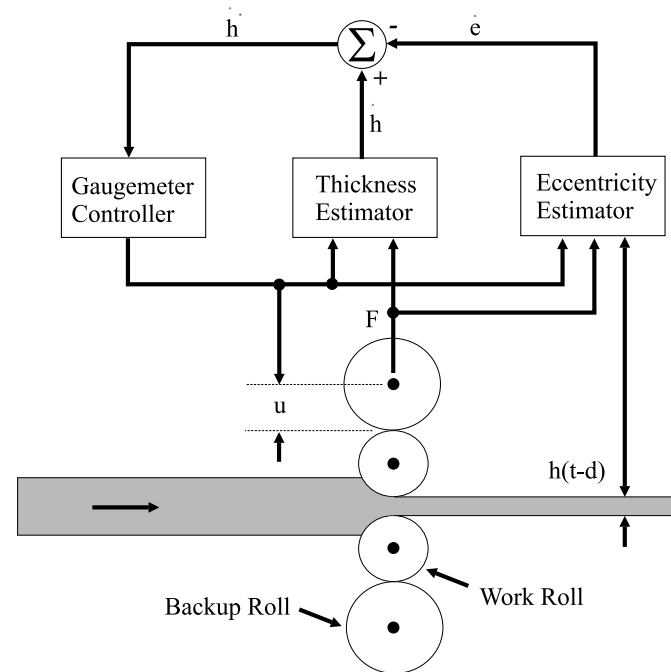
$$\{ x_1^k (t), x_2^k (t); \quad k = 1, \dots, N \}$$

and hence to correct the measured force measurements for eccentricity.

Note that this application has much in common with the satellite tracking problem since periodic functions are involved in both applications.

The final control system using the eccentricity compensated BISRA gauge is as shown on the next slide.

Figure 22.10: *Final roll eccentricity compensated control system*



An interesting feature of this problem is that there is some practical benefit in using the general time-varying form of the Kalman filter rather than the steady-state filter. The reason is that, in steady state, the filter acts as a narrow band-pass filter bank centred on the harmonic frequencies. This is, heuristically, the correct steady-state solution. However, an interesting fact that the reader can readily verify is that the transient response time of a narrow band-pass filter is inversely proportional to the filter bandwidth. This means that, in steady state, one has the following fundamental design trade-off:

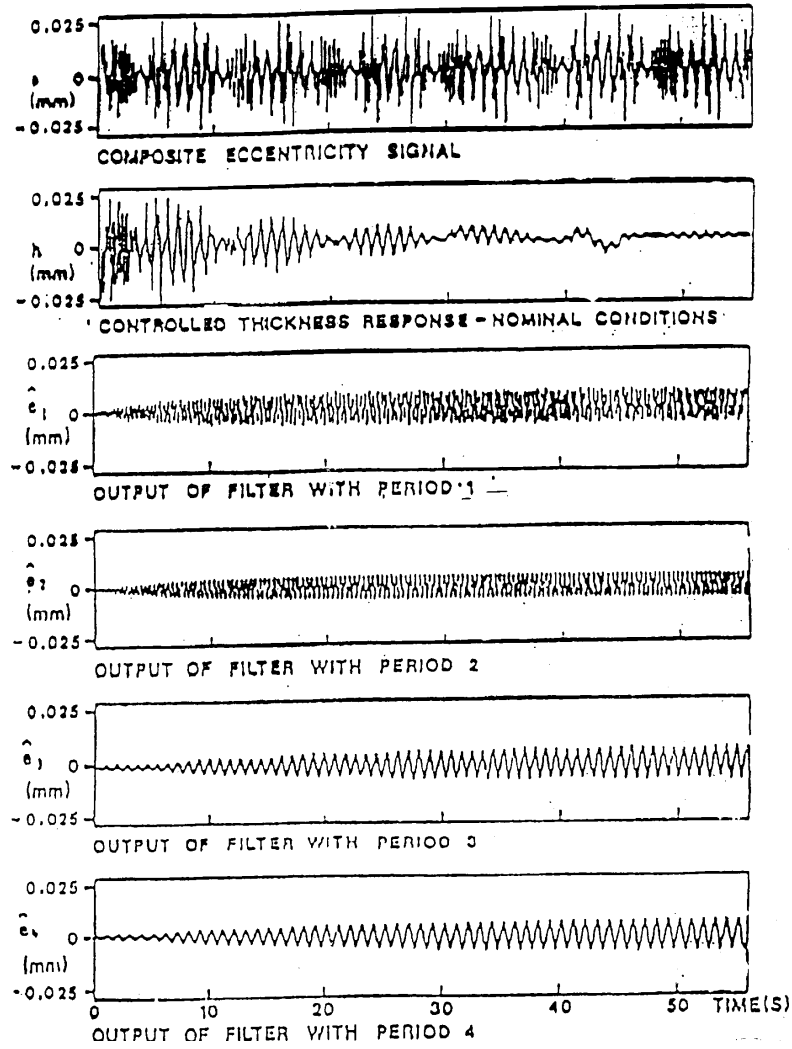
-
- ◆ On the one hand, one would like to have a narrow band-pass, to obtain good frequency selectivity and hence good noise rejection.
 - ◆ On the other hand, one would like to have a wide band-pass, to minimize the initial transient period.

This is an inescapable dichotomy for any *time-invariant* filter.

This suggests that one should not use a fixed filter gain but instead start with a wide-band filter, to minimize the transient, but then narrow the filter band down as the signal is acquired. This is precisely what the time-varying Kalman filter does.

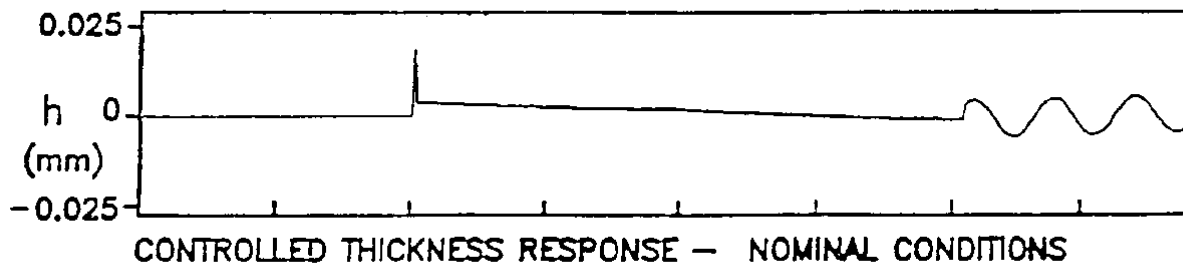
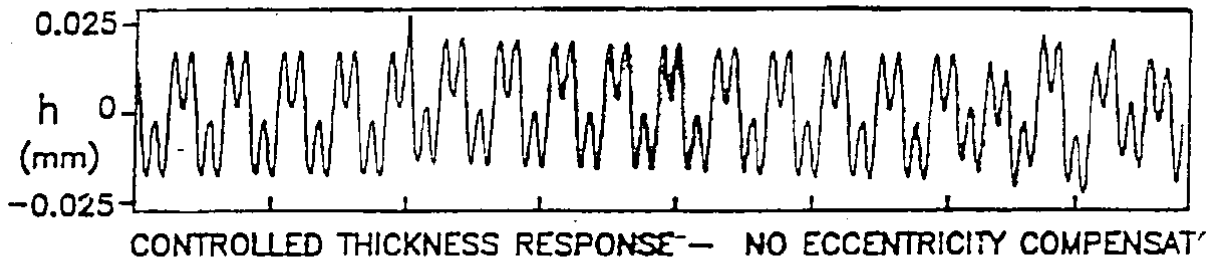
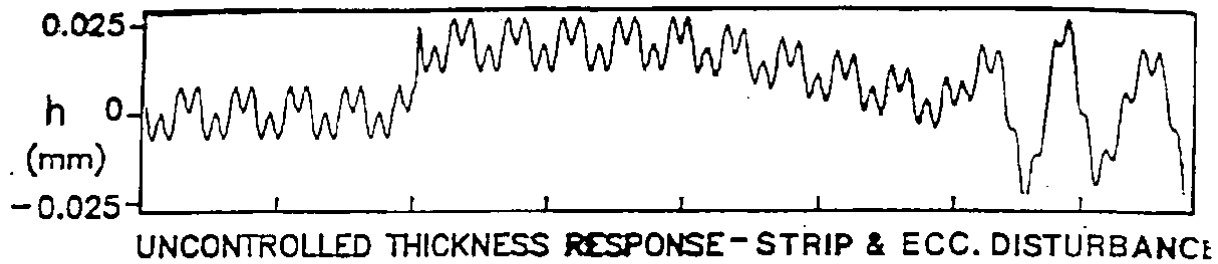
The next slide shows the efficacy of using the Kalman Filter to extract multiple sinusoidal components from a composite signal.

- ◆ The upper trace shows the composite signal which may look like random noise, but is in fact a combination of many sinewaves together with a noise component.
- ◆ The lower four traces show the extracted sinewaves corresponding to four of the frequencies. Note that after an initial transient the filter output settles to the sinewave component in the composite signal.



The next slide shows a simulation which demonstrates the advantages of using the Kalman Filter to compensate the BISRA gauge by removing the eccentricity components.

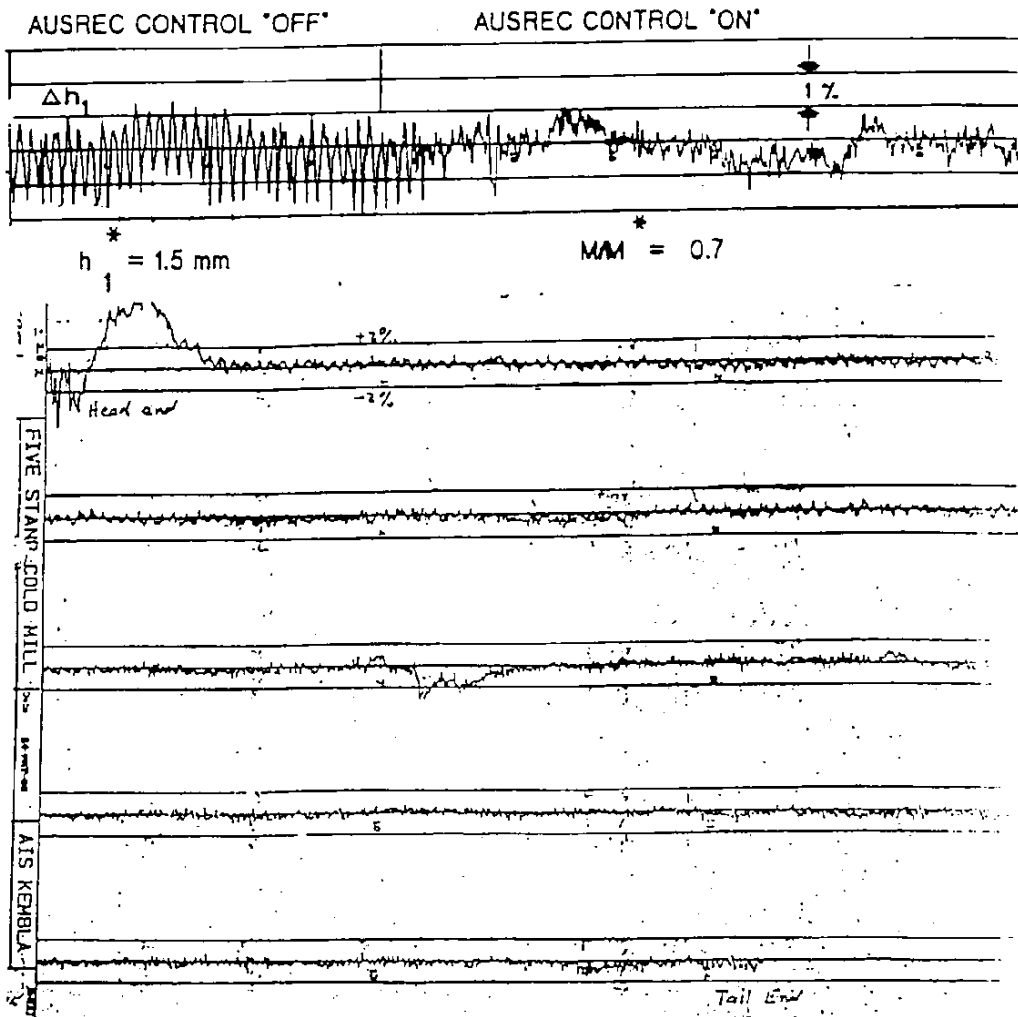
- ◆ The upper trace shows the uncontrolled response
- ◆ The middle trace shows the exit thickness response when a BISRA gauge is used but no eccentricity compensation is applied
- ◆ The lower trace shows the controlled exit thickness when the BISRA gauge is used for feedback having first been compensated using the Kalman Filter to remove the eccentricity components.



The next slide shows practical results of using eccentricity compensation on a practical rolling mill. The results were obtained on a tandem cold mill operated by BHP Steel International.

- ◆ The upper trace is divided into two halves. The left portion clearly shows the effect of eccentricity on the rolled thickness whilst the right hand portion shows the dramatic improvement resulting from using eccentricity compensation. Note that the drift in the mean on the right hand side is due to a different cause and can be readily rectified.

-
- ◆ The remainder of the traces show the effect of using an eccentricity compensated BISRA gauge on a full coil. The traces also show lines at $\pm 1\%$ error which was the design goal at the time these results were collected. Note that it is now common to have accuracies of $\pm 0.1\%$

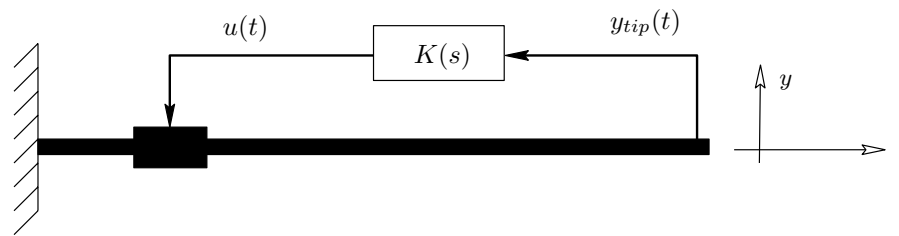


The final system, as described above, has been patented under the name AUSREC[®] and is available as a commercial product from Industrial Automation Services Pty. Ltd.

Vibration Control in Flexible Structures

Consider the problem of controller design for the piezoelectric laminate beam shown on the next slide.

Figure 22.11: *Vibration control by using a piezoelectric actuator*



This is a simple system. However, it represents many of the features of more complex systems where one wishes to control vibrations. Such problems occur in many problems, e.g. chatter in rolling mills, aircraft wing flutter, light weight space structures, etc.

In the laboratory system, the measurements are taken by a displacement sensor that is attached to the tip of the beam, and a piezoelectric patch is used as the actuator. The purpose of the controller is to minimize beam vibrations. It is easy to see that this is a regulator problem; hence, a LQG controller can be designed to reduce the unwanted vibrations.

To find the dynamics of structures such as the beam, one has to solve a particular partial differential equation that is known as the Bernoulli-Euler beam equation. By using modal analysis techniques, it is possible to show that a transfer function of the beam would consist of an infinite number of very lightly damped second-order resonant terms - that is, the transfer function from the voltage that is applied to the actuator to the displacement of the tip of the beam can be described by

$$G(s) = \sum_{i=1}^{\infty} \frac{\alpha_i}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}.$$

However, one is interested in designing a controller only for a particular bandwidth. As a result, it is common practice to truncate the model by keeping the first N modes that lie within the bandwidth of interest.

We consider a particular system and include only the first six modes of this system.

The transfer function is then

$$G(s) = \sum_{i=1}^6 \frac{\alpha_i}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}.$$

Here, ζ_i 's are assumed to be 0.002 and α_i 's as are shown in the Table below.

i	α_i	ω_i (rad/sec)
1	9.72×10^{-4}	18.95
2	0.0122	118.76
3	0.0012	332.54
4	-0.0583	651.660
5	-0.0013	1077.2
6	0.1199	1609.2

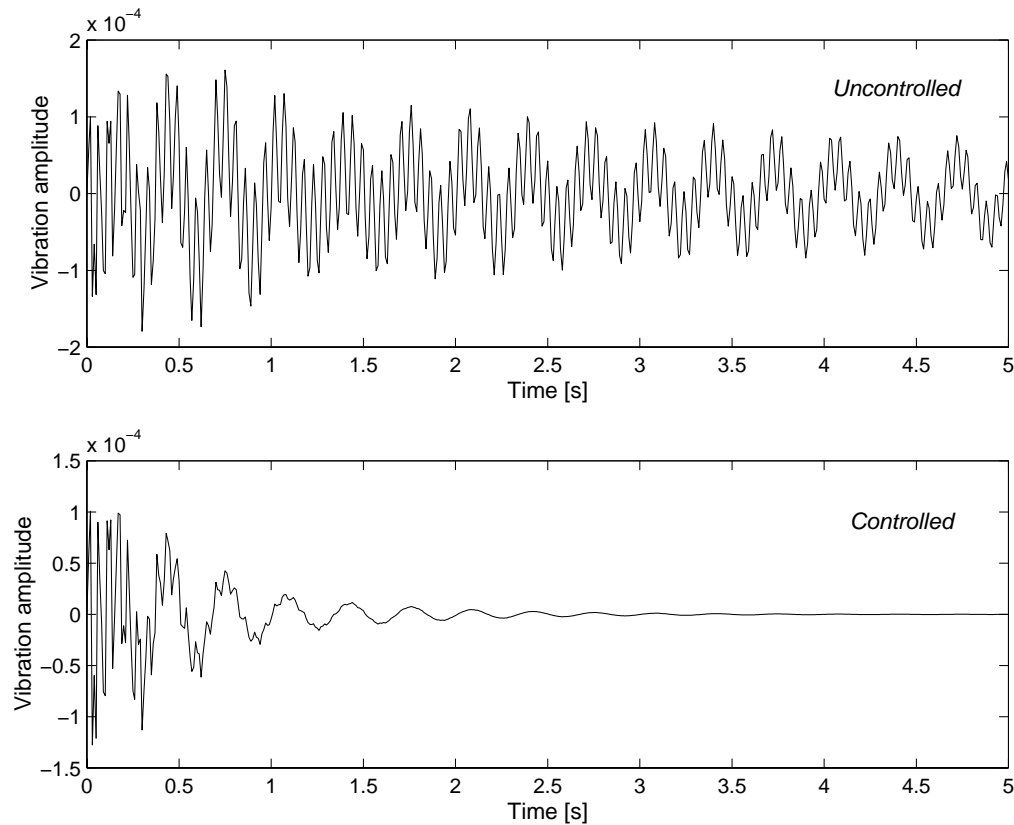
We design a Linear Quadratic Regulator. Here, the Ψ matrix is chosen to be

$$\Psi = 0.1483 \text{diag}(\omega_1^2, 1, \dots, \omega_6^2, 1)$$

The control-weighting matrix is also, somewhat arbitrarily, chosen as $\Phi = 10^{-8}$. Next, a Kalman-filter state estimator is designed with $\mathbf{Q} = 0.08I$ and $\mathbf{R} = 0.005$.

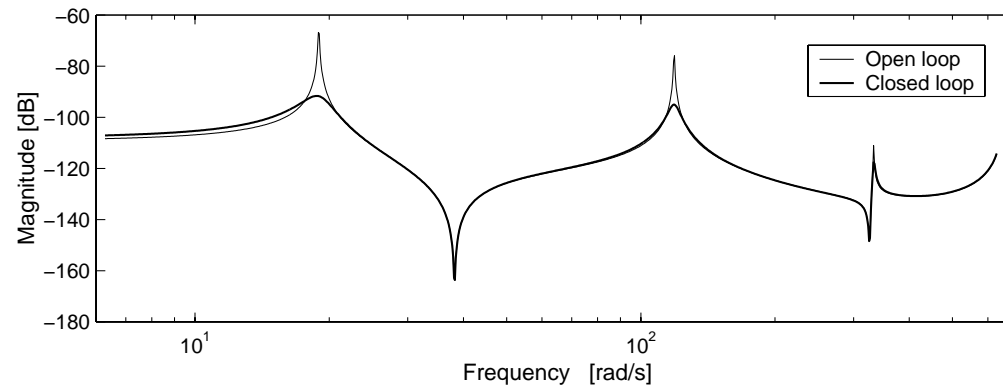
The next slide shows the simulated open-loop and closed-loop impulse responses of the system. It can be observed that the LQG controller can considerably reduce structural vibrations.

Figure 22.12: *Open-loop and closed-loop impulse responses of the beam*



On the next slide we show the open-loop and closed-loop frequency responses of the beam. It can be observed that the LQG controller has significantly damped the first three resonant modes of the structure.

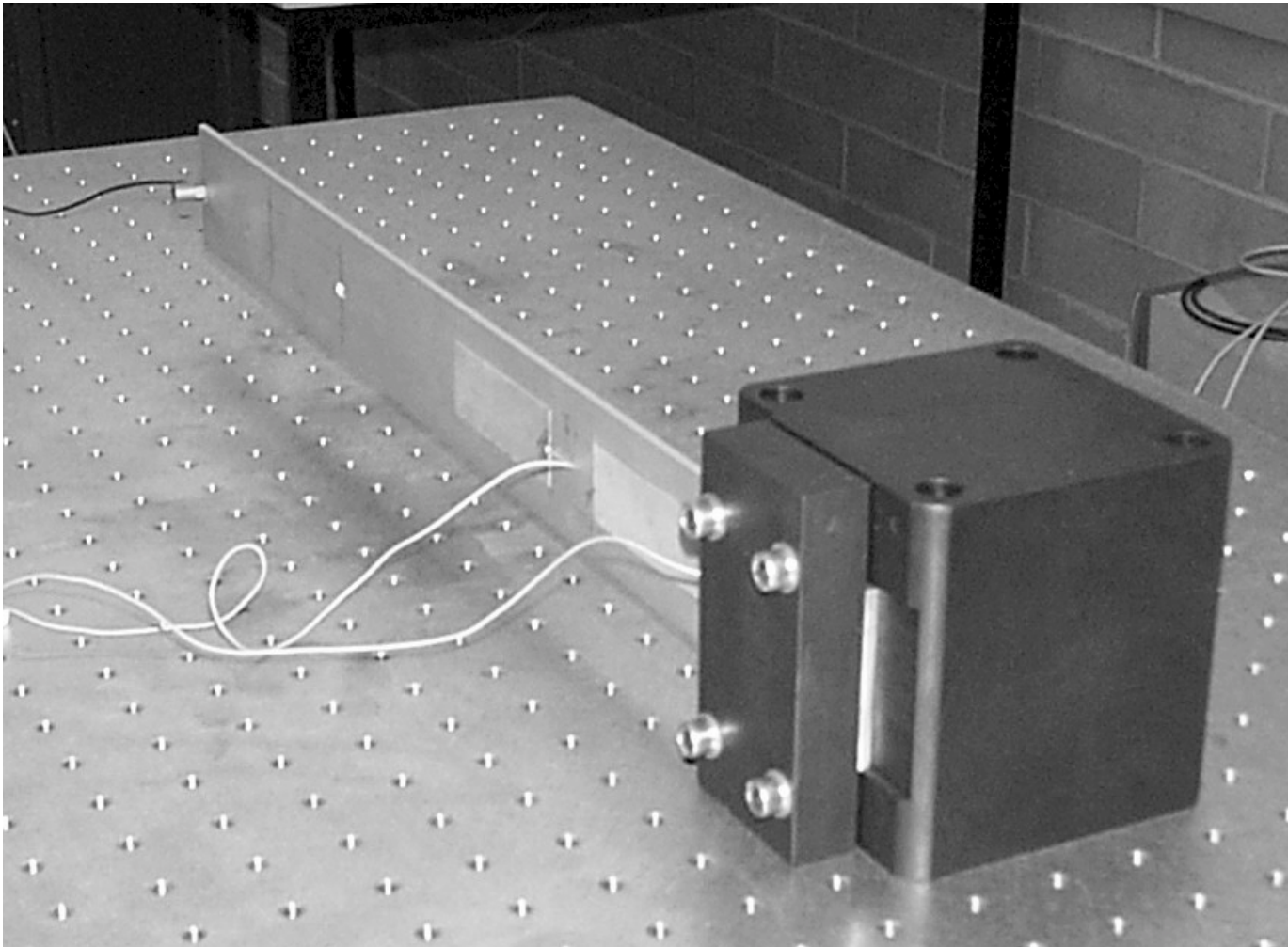
Figure 22.13: *Open-loop and closed-loop frequency responses of the beam*



Experimental Apparatus

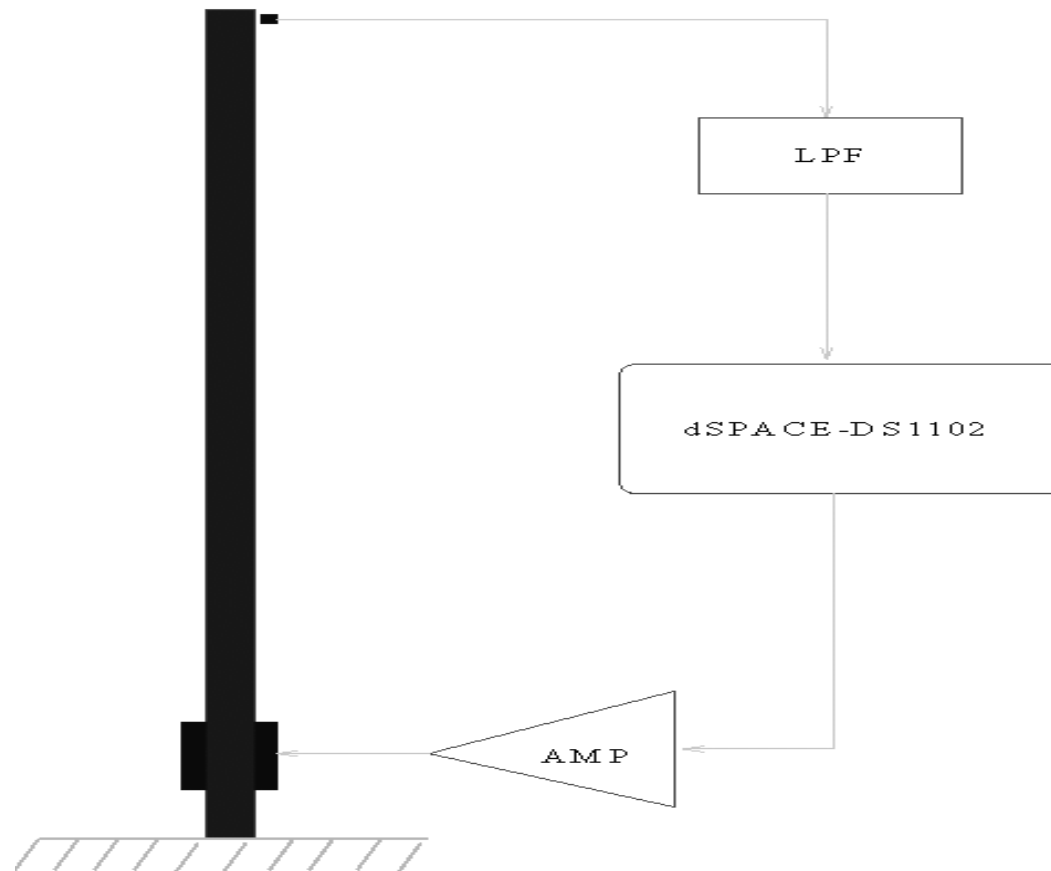
A photograph of an experimental rig (*at the University of Newcastle Australia*) of a flexible beam used to study vibration control is shown on the next slide.

Experimental Rig of Flexible Beam



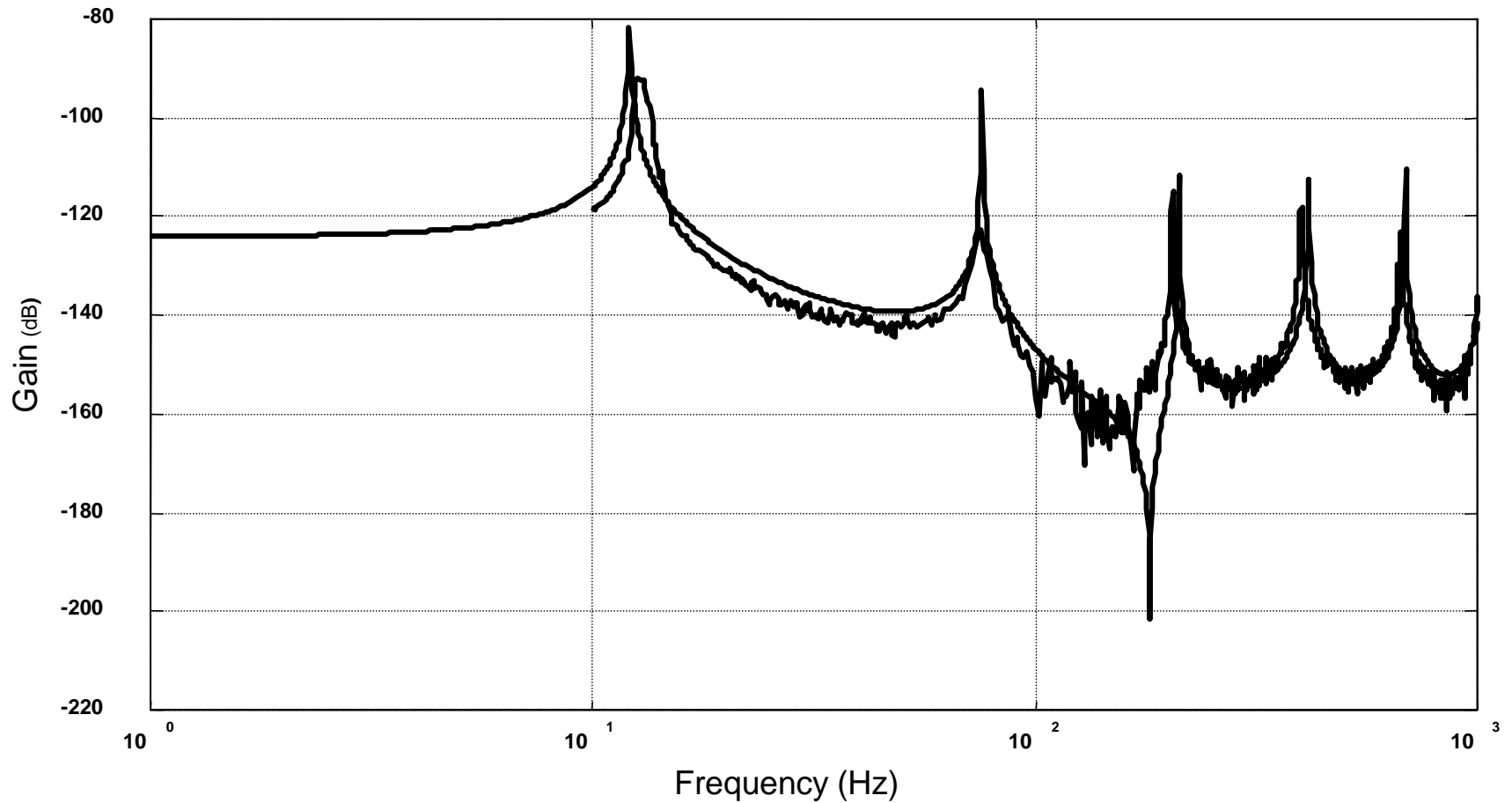
A schematic of the beam including the controller
(*which is here implemented in a dSpace[®] controller*)
is shown on the next slide.

Schematic of Experimental Set Up

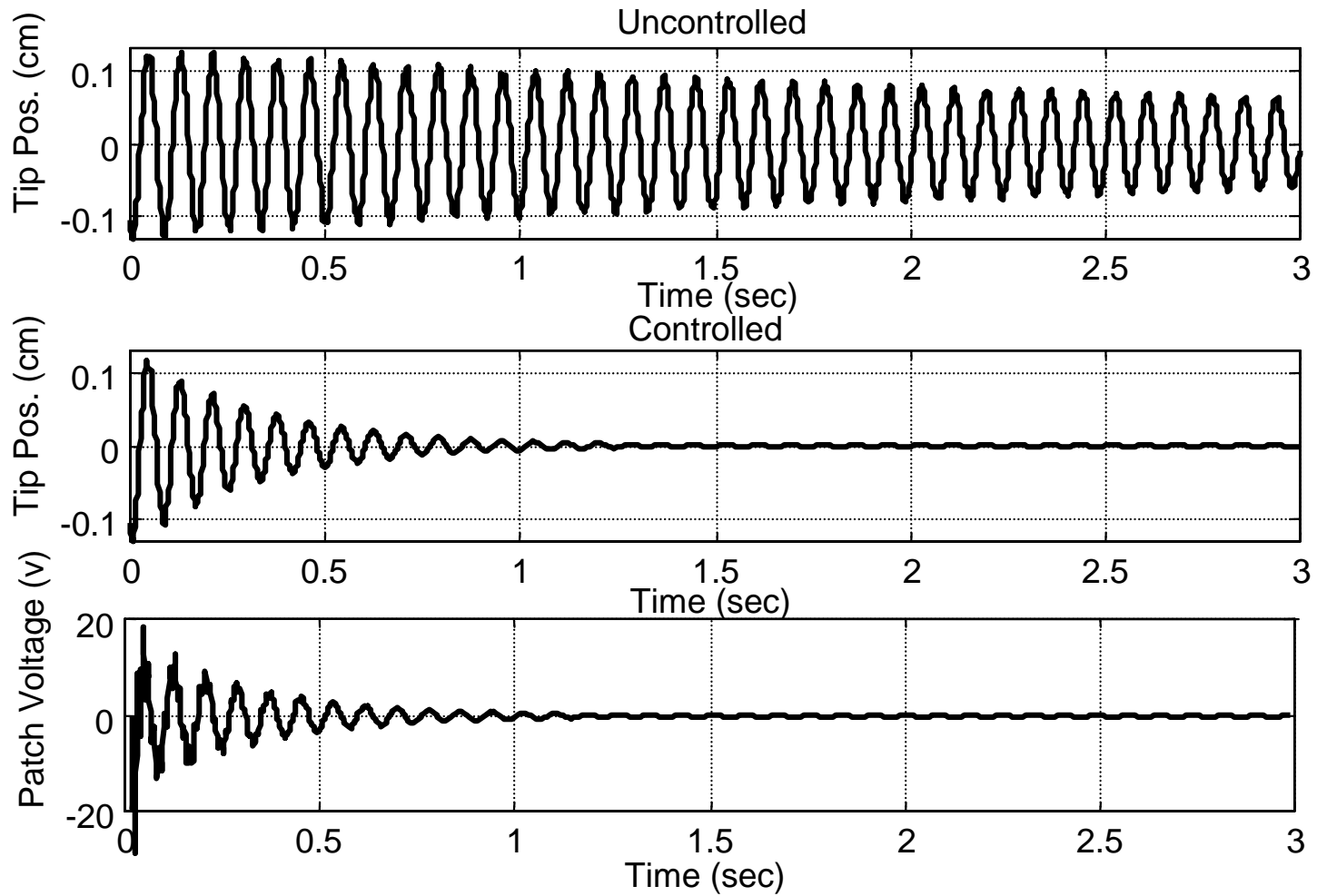


The experimentally measured frequency response is shown on the next slide - note that the system is highly resonant as predicted in the model described earlier. (*The smooth line corresponds to the model*).

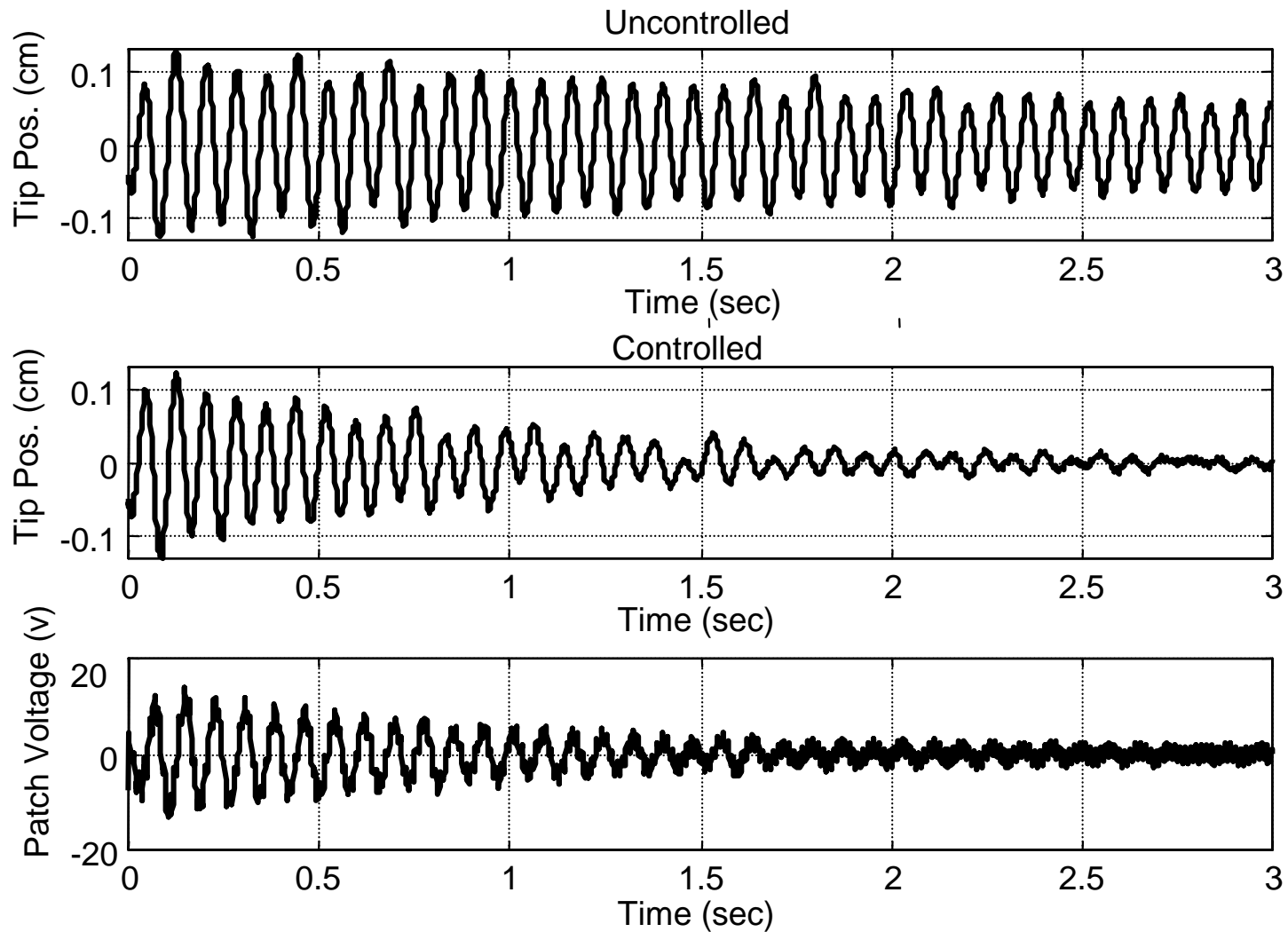
Frequency Responses



The next slide shows simulated responses following an impulsive type disturbance applied to the system.

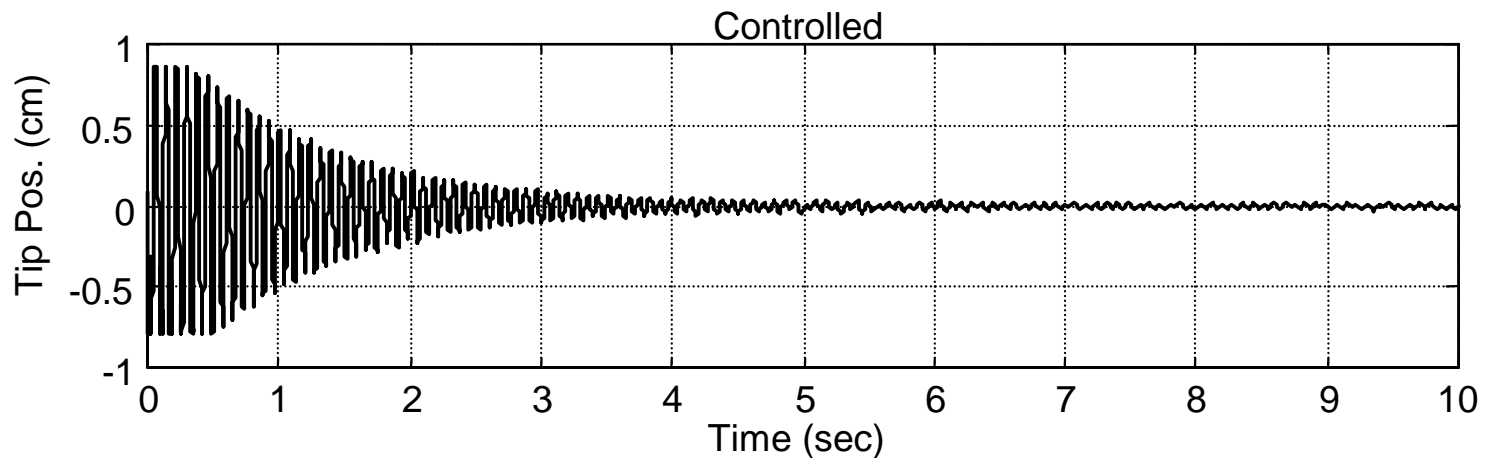
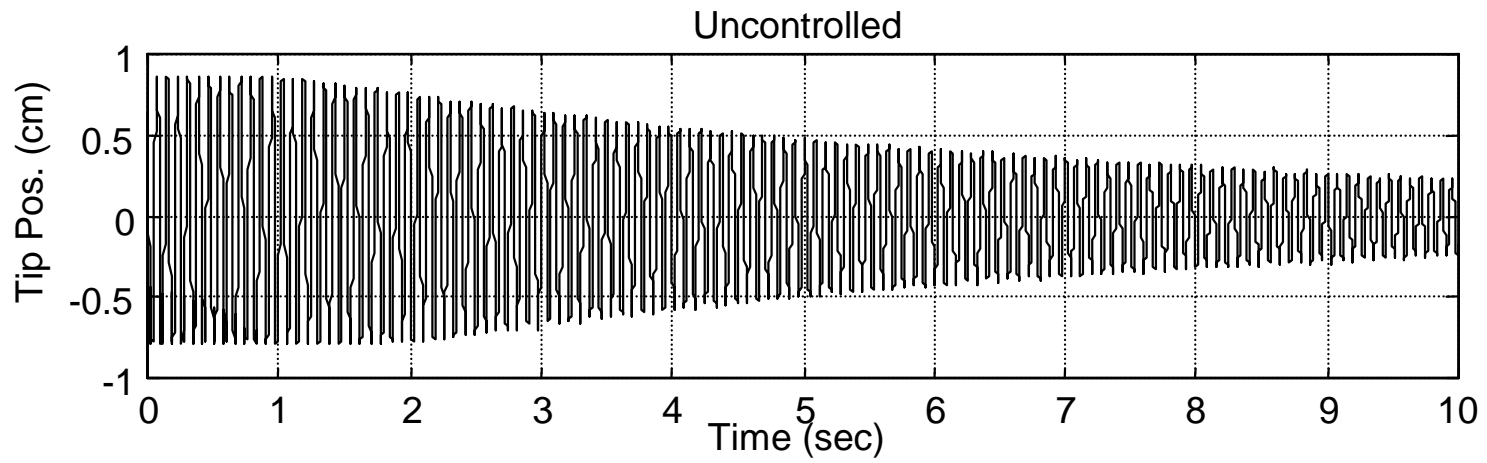


The next slide shows experimental results using the dSpace based controller with an impulsive type input.



The next slide compares the exponential response to an impulse type disturbance with and without closed loop control.

Experimental Results: Free and Controlled Responses



Of course, the above experimental set-up is quite simple. However, as we have said earlier, similar problems occur in many real world problems. By way of illustrating, the next slide shows the tail-fin of an FA-18 fighter aircraft. This tail fin has been covered in piezoelectric patches. These patches are aimed at reducing vibrations arising from turbulence.

FA-18 Tail



Summary

- ❖ Full multivariable control incorporates the interaction dynamics rigorously and explicitly.
- ❖ The fundamental SISO synthesis result that, under mild conditions, the nominal closed-loop poles can be assigned arbitrarily carries over to the MIMO case.
- ❖ Equivalence of state-feedback and frequency-domain pole placement by solving the (*multivariable*) Diophantine Equation carries over as well.
- ❖ The complexities of multivariable systems, cause criterion-based synthesis (*briefly alluded to in the SISO case*) to gain additional motivation.

-
- ❖ A popular family of criteria are functionals involving quadratic forms of control error and control effort.
 - ❖ For a general nonlinear formulation, the optimal solution is characterized by a two-point boundary-value problem.
 - ❖ In the linear case (the so-called linear quadratic regulator, LQR) the general problem reduces to the solution of the continuous-time dynamic Riccati equation, which can be feasibly solved, leading to time-variable state feedback.
 - ❖ After initial conditions decay, the optimal time-varying solution converges to a constant state feedback, the so-called steady-state LQR solution.

-
- ❖ It is frequently sufficient to neglect the initial transient of the strict LQR and only implement the steady-state LQR.
 - ❖ The steady-state LQR is equivalent
 - ◆ to a model-matching approach, where a desired complementary sensitivity is specified and a controller is computer that matches it as closely as possible according to some selected measure, and
 - ◆ to pole placement, where a closed-loop polynomial is specified and a controller is computed to achieve it.

-
- ❖ Thus, LQR, model matching, and pole placement are mathematically equivalent, although they do offer different tuning parameters.

<i>Equivalent synthesis techniques</i>	<i>Tuning parameters</i>
LQR	relative penalties on control error versus control effort.
Model matching	closed-loop complementary sensitivity reference model and weighted penalty on the difference to the control loop.
Pole placement	closed-loop polynomial

-
- ❖ These techniques can be extended to discrete-time systems.
 - ❖ There is a very close connection to the dual problem of filtering: inferring a state from a related (*but not exactly invertible*) set of measurements.
 - ❖ Optimal-filter design based on quadratic criteria leads again to a Riccati equation.
 - ❖ The filters can be synthesized and interpreted equivalently in a
 - ◆ linear quadratic,
 - ◆ model-matching, or
 - ◆ pole-placementframework.

-
- ❖ The arguably most famous optimal-filter formulation, the Kalman filter, can be given a stochastic or a deterministic interpretation, depending on taste.
 - ❖ The LQR does not automatically include integral action; thus, rejection of constant or other polynomial disturbances must be enforced via the Internal Model Principle.